

ANDÔ DILATIONS AND INEQUALITIES ON NONCOMMUTATIVE DOMAINS

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ABSTRACT. We obtain intertwining dilation theorems for noncommutative regular domains \mathcal{D}_f and noncommutative varieties \mathcal{V}_J in $B(\mathcal{H})^n$, which generalize Sarason [18] and Sz.-Nagy–Foiaş [20] commutant lifting theorem for commuting contractions. We present several applications including a new proof for the commutant lifting theorem for pure elements in the domain \mathcal{D}_f (resp. variety \mathcal{V}_J) as well as a Schur type representation for the unit ball of the Hardy algebra associated with the variety \mathcal{V}_J .

We provide Andô type dilations and inequalities for bi-domains $\mathcal{D}_f \times_c \mathcal{D}_g$ which consist of all pairs (\mathbf{X}, \mathbf{Y}) of tuples $\mathbf{X} := (X_1, \dots, X_{n_1}) \in \mathcal{D}_f$ and $\mathbf{Y} := (Y_1, \dots, Y_{n_2}) \in \mathcal{D}_g$ which commute, i.e. each entry of \mathbf{X} commutes with each entry of \mathbf{Y} . The results are new even when $n_1 = n_2 = 1$. In this particular case, we obtain extensions of Andô's results [2] and Agler-McCarthy's inequality [1] for commuting contractions to larger classes of commuting operators.

All the results are extended to bi-varieties $\mathcal{V}_{J_1} \times_c \mathcal{V}_{J_2}$, where \mathcal{V}_{J_1} and \mathcal{V}_{J_2} are noncommutative varieties generated by WOT-closed two-sided ideals in noncommutative Hardy algebras. The commutative case as well as the matrix case when $n_1 = n_2 = 1$ are also discussed.

INTRODUCTION

Extending von Neumann [23] inequality for one contraction and Sz.-Nagy proof using dilation theory [21], Andô [2] proved a dilation result that implies his celebrated inequality which says that if T_1 and T_2 are commuting contractions on a Hilbert space, then for any polynomial p in two variables,

$$\|p(T_1, T_2)\| \leq \|p\|_{\mathbb{D}^2},$$

where \mathbb{D}^2 is the bidisk in \mathbb{C}^2 . For a nice survey and further generalizations of these inequalities we refer to Pisier's book [9] (see also [23], [21], [7], [22], [10], [8], [4], [5], [1], and [6]). Inspired by the work of Agler-McCarthy [1] and Das-Sarkar [6] on distinguished varieties and Andô's inequality for two commuting contractions, we found, in a very recent paper [16], analogues of Andô's results for the elements of the bi-ball \mathbf{P}_{n_1, n_2} which consists of all pairs (\mathbf{X}, \mathbf{Y}) of row contractions $\mathbf{X} := (X_1, \dots, X_{n_1})$ and $\mathbf{Y} := (Y_1, \dots, Y_{n_2})$ which commute, i.e. each entry of \mathbf{X} commutes with each entry of \mathbf{Y} . The results were obtained in a more general setting, namely, when \mathbf{X} and \mathbf{Y} belong to noncommutative varieties \mathcal{V}_1 and \mathcal{V}_2 determined by row contractions subject to constraints such as

$$q(X_1, \dots, X_{n_1}) = 0 \quad \text{and} \quad r(Y_1, \dots, Y_{n_2}) = 0, \quad q \in \mathcal{P}, r \in \mathcal{R},$$

respectively, where \mathcal{P} and \mathcal{R} are sets of noncommutative polynomials. This led to one of the main results of the paper, an Andô type inequality on noncommutative varieties, which, in the particular case when $n_1 = n_2 = 1$ and T_1 and T_2 are commuting contractive matrices with spectrum in the open unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$, takes the form

$$\|p(T_1, T_2)\| \leq \min \{ \|p(B_1 \otimes I_{\mathbb{C}^{d_1}}, \varphi_1(B_1))\|, \|p(\varphi_2(B_2), B_2 \otimes I_{\mathbb{C}^{d_2}})\| \},$$

where $(B_1 \otimes I_{\mathbb{C}^{d_1}}, \varphi_1(B_1))$ and $(\varphi_2(B_2), B_2 \otimes I_{\mathbb{C}^{d_2}})$ are analytic dilations of (T_1, T_2) while B_1 and B_2 are the universal models associated with T_1 and T_2 , respectively. In this setting, the inequality is sharper than Andô's inequality and Agler-McCarthy's inequality [1]. We obtained more general inequalities for arbitrary commuting contractive matrices and improve Andô's inequality for commuting contractions

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when at least one of them is of class \mathcal{C}_0 . In this setting, it would be interesting to find good analogues for *distinguished varieties* in the sense of [1].

Let \mathbb{F}_n^+ be the unital free semigroup on n generators g_1, \dots, g_n and the identity g_0 . The length of $\alpha \in \mathbb{F}_n^+$ is defined by $|\alpha| := 0$ if $\alpha = g_0$ and $|\alpha| := k$ if $\alpha = g_{i_1} \cdots g_{i_k}$, where $i_1, \dots, i_k \in \{1, \dots, n\}$. If $\mathbf{Z} := \langle Z_1, \dots, Z_n \rangle$ is an n -tuple of noncommutative indeterminates, we use the notation $Z_\alpha := Z_{i_1} \cdots Z_{i_k}$ and $Z_{g_0} := 1$. We denote by $\mathbb{C}\langle \mathbf{Z} \rangle$ the complex algebra of all polynomials in Z_1, \dots, Z_n . A polynomial $f := \sum_{\alpha \in \mathbb{F}_n^+} a_\alpha Z_\alpha$ in $\mathbb{C}\langle \mathbf{Z} \rangle$ is called *positive regular* if the coefficients satisfy the conditions: $a_\alpha \geq 0$ for any $\alpha \in \mathbb{F}_n^+$, $a_{g_0} = 0$, and $a_{g_i} > 0$ if $i = 1, \dots, n$. Define the *noncommutative regular domain*

$$\mathcal{D}_f(\mathcal{H}) := \left\{ \mathbf{X} := (X_1, \dots, X_n) \in B(\mathcal{H})^n : \sum_{|\alpha| \geq 1} a_\alpha X_\alpha X_\alpha^* \leq I_{\mathcal{H}} \right\}$$

and the *noncommutative ellipsoid* $\mathcal{E}_f(\mathcal{H}) \supseteq \mathcal{D}_f(\mathcal{H})$ by setting

$$\mathcal{E}_f(\mathcal{H}) := \left\{ \mathbf{X} := (X_1, \dots, X_n) \in B(\mathcal{H})^n : \sum_{|\beta|=1} a_\beta X_\beta X_\beta^* \leq I_{\mathcal{H}} \right\},$$

where $B(\mathcal{H})$ stands for the algebra of all bounded linear operators on a Hilbert space \mathcal{H} . Given $n_1, n_2 \in \mathbb{N} := \{1, 2, \dots\}$ and $\Omega_j \subseteq B(\mathcal{H})^{n_j}$, $j = 1, 2$, we denote by $\Omega_1 \times_c \Omega_2$ the set of all pairs $(\mathbf{X}, \mathbf{Y}) \in \Omega_1 \times \Omega_2$ with the property that the entries of $\mathbf{X} := (X_1, \dots, X_{n_1})$ are commuting with the entries of $\mathbf{Y} := (Y_1, \dots, Y_{n_2})$.

The main goal of the present paper is to extend the results from [16] for bi-balls and obtain Andô type dilations and inequalities for bi-domains and noncommutative varieties:

$$\mathcal{D}_f(\mathcal{H}) \times_c \mathcal{D}_g(\mathcal{H}) \quad \text{and} \quad \mathcal{V}_{J_1}(\mathcal{H}) \times_c \mathcal{V}_{J_2}(\mathcal{H}),$$

where $f \in \mathbb{C}\langle \mathbf{Z} \rangle$ and $g \in \mathbb{C}\langle \mathbf{Z}' \rangle$ are positive regular noncommutative polynomials while \mathcal{V}_{J_1} and \mathcal{V}_{J_2} are varieties generated by WOT-closed two-sided ideals in certain noncommutative Hardy algebras.

In Section 2, we obtain an intertwining dilation theorem for bi-domains which generalizes Sarason [18] and Sz.-Nagy–Foiaş [20] commutant lifting theorem for commuting contractions in the framework of noncommutative regular domains and Poisson kernels on weighted Fock spaces (see [15]). As a consequence, we obtain a new proof for the commutant lifting theorem for pure elements in \mathcal{D}_f .

These results are extended, in Section 3, to noncommutative varieties $\mathcal{V}_J \subseteq \mathcal{D}_f$ which are generated by WOT-closed two-sided ideals J in the Hardy algebra $F_n^\infty(\mathcal{D}_f)$, a noncommutative multivariable version of the classical Hardy algebra $H^\infty(\mathbb{D})$. More precisely, the noncommutative variety $\mathcal{V}_J(\mathcal{H})$ is defined as the set of all *pure* n -tuples $\mathbf{X} := (X_1, \dots, X_n) \in \mathcal{D}_f(\mathcal{H})$ with the property that

$$\varphi(X_1, \dots, X_n) = 0 \quad \text{for any } \varphi \in J,$$

where $\varphi(X_1, \dots, X_n)$ is defined using the $F_n^\infty(\mathcal{D}_f)$ -functional calculus for pure elements in $\mathcal{D}_f(\mathcal{H})$ (see [15]). Each variety \mathcal{V}_J is associated with certain universal models $\mathbf{B} = (B_1, \dots, B_n)$ and $\mathbf{C} = (C_1, \dots, C_n)$ of constrained creation operators acting on a subspace \mathcal{N}_J of the full Fock space with n generators $F^2(H_n)$. The noncommutative Hardy algebras $F_n^\infty(\mathcal{V}_J)$ and $R_n^\infty(\mathcal{V}_J)$ are the WOT-closed algebras generated by I, B_1, \dots, B_n and I, C_1, \dots, C_n , respectively. Using our intertwining dilation theorem on noncommutative varieties, we obtain a Schur [19] type representation for the unit ball of $\mathcal{R}_n^\infty(\mathcal{V}_J) \bar{\otimes} B(\mathcal{H}', \mathcal{H})$.

In Section 4, we obtain Andô type dilations and inequalities for noncommutative varieties

$$\mathcal{V}_{J_1}(\mathcal{H}) \times_c \mathcal{V}_{J_2}(\mathcal{H}),$$

where $\mathcal{V}_{J_1}(\mathcal{H}) \subseteq \mathcal{D}_f(\mathcal{H})$ and $\mathcal{V}_{J_2}(\mathcal{H}) \subseteq \mathcal{D}_g(\mathcal{H})$. We prove that any pair $(\mathbf{T}_1, \mathbf{T}_2)$ in $\mathcal{V}_{J_1}(\mathcal{H}) \times_c \mathcal{V}_{J_2}(\mathcal{H})$ has *analytic dilations*

$$(\mathbf{B}_1 \otimes I_{\ell^2}, \varphi_1(\mathbf{C}_1)) \quad \text{and} \quad (\varphi_2(\mathbf{C}_2), \mathbf{B}_2 \otimes I_{\ell^2})$$

where $\varphi_1(\mathbf{C}_1)$ and $\varphi_2(\mathbf{C}_2)$ are some multi-analytic operators with respect to the universal models \mathbf{B}_1 and \mathbf{B}_2 of the varieties \mathcal{V}_{J_1} and \mathcal{V}_{J_2} , respectively. As a consequence, we show that the inequality

$$\| [p_{rs}(\mathbf{T}_1, \mathbf{T}_2)]_k \| \leq \min \{ \| [p_{rs}(\mathbf{B}_1 \otimes I_{\ell^2}, \varphi_1(\mathbf{C}_1))]_k \|, \| [p_{rs}(\varphi_2(\mathbf{C}_2), \mathbf{B}_2 \otimes I_{\ell^2})]_k \| \}$$

holds for any $[p_{rs}]_k \in M_k(\mathbb{C}\langle \mathbf{Z}, \mathbf{Z}' \rangle)$ and $k \in \mathbb{N}$. Here, $\mathbb{C}\langle \mathbf{Z}, \mathbf{Z}' \rangle$ denotes the complex algebra of all polynomials in noncommutative indeterminates $\mathbf{Z} := \langle Z_1, \dots, Z_{n_1} \rangle$ and $\mathbf{Z}' := \langle Z'_1, \dots, Z'_{n_2} \rangle$, where we assume that $Z_i Z'_j = Z'_j Z_i$ for any $i \in \{1, \dots, n_1\}$ and $j \in \{1, \dots, n_2\}$.

On the other hand, we prove that the abstract bi-domain

$$\mathcal{D}_f \times_c \mathcal{E}_g := \{\mathcal{D}_f(\mathcal{H}) \times_c \mathcal{E}_g(\mathcal{H}) : \mathcal{H} \text{ is a Hilbert space}\}$$

has a universal *analytic model* $(\mathbf{W}_1 \otimes I_{\ell^2}, \psi(\mathbf{\Lambda}_1))$, where the tuples $\mathbf{W}_1 = (W_{1,1}, \dots, W_{1,n_1})$ and $\mathbf{\Lambda}_1 = (\Lambda_{1,1}, \dots, \Lambda_{1,n_1})$ are the weighted left and right creation operators on the full Fock space $F^2(H_{n_1})$, respectively, associated with the regular domain \mathcal{D}_f . More precisely, we show that the inequality

$$\|[p_{rs}(\mathbf{T}_1, \mathbf{T}_2)]_k\| \leq \|[p_{rs}(\mathbf{W}_1 \otimes I_{\ell^2}, \psi(\mathbf{\Lambda}_1))]_k\|, \quad [p_{rs}]_k \in M_k(\mathbb{C}\langle \mathbf{Z}, \mathbf{Z}' \rangle),$$

holds for any $(\mathbf{T}_1, \mathbf{T}_2) \in \mathcal{D}_f(\mathcal{H}) \times_c \mathcal{E}_g(\mathcal{H})$ and any $k \in \mathbb{N}$. A similar result holds for the abstract variety $\mathcal{V}_J \times_c \mathcal{E}_g$.

We will see, in Section 4, that all the results of the present paper concerning Andô type dilations and inequalities can be written in the commutative multivariable setting in terms of analytic multipliers of certain Hilbert spaces of holomorphic functions. These results are new even when $n_1 = n_2 = 1$. In this particular case, we obtain extensions of Andô's results for commuting contractions [2], Agler-McCarthy's inequality [1], and Das-Sarkar extension [6], to larger classes of commuting operators.

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1. PRELIMINARIES ON NONCOMMUTATIVE REGULAR DOMAINS AND UNIVERSAL MODELS

In this section, we recall from [15] basic facts concerning the noncommutative regular domains $\mathcal{D}_f(\mathcal{H}) \subset B(\mathcal{H})^n$ generated by positive regular formal power series, their universal models, and the Hardy algebras they generate. We mention that commutative domains generated by positive regular polynomials were first introduced in [17] and further elaborated in [3] and in a series of papers by the author (see [15] and the references there in).

Let H_n be an n -dimensional complex Hilbert space with orthonormal basis e_1, e_2, \dots, e_n , where $n \in \{1, 2, \dots\}$. We consider the full Fock space of H_n defined by

$$F^2(H_n) := \bigoplus_{k \geq 0} H_n^{\otimes k},$$

where $H_n^{\otimes 0} := \mathbb{C}1$ and $H_n^{\otimes k}$ is the (Hilbert) tensor product of k copies of H_n . Define the left creation operators $S_i : F^2(H_n) \rightarrow F^2(H_n)$, $i = 1, \dots, n$, by

$$S_i \varphi := e_i \otimes \varphi, \quad \varphi \in F^2(H_n),$$

and the right creation operators $R_i : F^2(H_n) \rightarrow F^2(H_n)$ by $R_i \varphi := \varphi \otimes e_i$, $\varphi \in F^2(H_n)$. The noncommutative analytic Toeplitz algebra F_n^∞ and its norm closed version, the noncommutative disc algebra \mathcal{A}_n , were introduced by the author (see [10], [11], [12]) in connection with a multivariable noncommutative von Neumann inequality. F_n^∞ is the algebra of left multipliers of $F^2(H_n)$ and can be identified with the weakly closed (or w^* -closed) algebra generated by the left creation operators S_1, \dots, S_n acting on $F^2(H_n)$, and the identity. The noncommutative disc algebra \mathcal{A}_n is the norm closed algebra generated by S_1, \dots, S_n , and the identity.

A formal power series $f := \sum_{\alpha \in \mathbb{F}_n^+} a_\alpha Z_\alpha$ in noncommutative indeterminates Z_1, \dots, Z_n is called *positive regular* if the coefficients satisfy the conditions: $a_\alpha \geq 0$ for any $\alpha \in \mathbb{F}_n^+$, $a_{g_0} = 0$, $a_{g_i} > 0$ if $i = 1, \dots, n$, and $\limsup_{k \rightarrow \infty} \left(\sum_{|\alpha|=k} |a_\alpha|^2 \right)^{1/2k} < \infty$. If $\mathbf{X} := (X_1, \dots, X_n) \in B(\mathcal{H})^n$, we set $X_\alpha := X_{i_1} \cdots X_{i_k}$ if $\alpha = g_{i_1} \cdots g_{i_k}$, where $i_1, \dots, i_k \in \{1, \dots, n\}$, and $X_{g_0} := I_{\mathcal{H}}$. Define the noncommutative regular domain

$$\mathcal{D}_f(\mathcal{H}) := \left\{ \mathbf{X} := (X_1, \dots, X_n) \in B(\mathcal{H})^n : \sum_{|\alpha| \geq 1} a_\alpha X_\alpha X_\alpha^* \leq I_{\mathcal{H}} \right\},$$

where the convergence of the series is in the weak operator topology. The power series $1 - f$ is invertible with its inverse $g = \sum_{\alpha \in \mathbb{F}_n^+} b_\alpha X_\alpha$, $b_\alpha \in \mathbb{C}$, satisfies the relation

$$g = 1 + f + f^2 + \cdots = 1 + \sum_{m=1}^{\infty} \sum_{|\alpha|=m} \left(\sum_{j=1}^{|\alpha|} \sum_{\substack{\gamma_1 \cdots \gamma_j = \alpha \\ |\gamma_1| \geq 1, \dots, |\gamma_j| \geq 1}} a_{\gamma_1} \cdots a_{\gamma_j} \right) X_\alpha.$$

Consequently, we have

$$(1.1) \quad b_{g_0} = 1 \quad \text{and} \quad b_\alpha = \sum_{j=1}^{|\alpha|} \sum_{\substack{\gamma_1 \cdots \gamma_j = \alpha \\ |\gamma_1| \geq 1, \dots, |\gamma_j| \geq 1}} a_{\gamma_1} \cdots a_{\gamma_j} \quad \text{if } |\alpha| \geq 1.$$

The *weighted left creation operators* $W_i : F^2(H_n) \rightarrow F^2(H_n)$, $i = 1, \dots, n$, associated with the noncommutative domain \mathcal{D}_f are defined by setting $W_i := S_i D_i$, where S_1, \dots, S_n are the left creation operators on the full Fock space $F^2(H_n)$ and each diagonal operator $D_i : F^2(H_n) \rightarrow F^2(H_n)$, is given by

$$D_i e_\alpha := \sqrt{\frac{b_\alpha}{b_{g_i \alpha}}} e_\alpha, \quad \alpha \in \mathbb{F}_n^+.$$

Note that

$$W_\beta e_\gamma = \frac{\sqrt{b_\gamma}}{\sqrt{b_{\beta\gamma}}} e_{\beta\gamma} \quad \text{and} \quad W_\beta^* e_\alpha = \begin{cases} \frac{\sqrt{b_\gamma}}{\sqrt{b_\alpha}} e_\gamma & \text{if } \alpha = \beta\gamma \\ 0 & \text{otherwise} \end{cases}$$

for any $\alpha, \beta \in \mathbb{F}_n^+$. We prove in [15] that $\mathbf{W} := (W_1, \dots, W_n)$ is a pure n -tuple in $\mathcal{D}_f(F^2(H_n))$, i.e. $\Phi_{f, \mathbf{W}}(I) \leq I$ and $\Phi_{f, \mathbf{W}}^k(I) \rightarrow 0$ in the strong operator topology, as $k \rightarrow \infty$, where $\Phi_{f, \mathbf{W}}(Y) := \sum_{|\alpha| \geq 1} a_\alpha W_\alpha Y W_\alpha^*$ for $Y \in B(F^2(H_n))$ and the convergence is in the weak operator topology. The n -tuple

\mathbf{W} plays the role of universal model for the noncommutative domain \mathcal{D}_f .

We also define the *weighted right creation operators* $\Lambda_i : F^2(H_n) \rightarrow F^2(H_n)$ by setting $\Lambda_i := R_i G_i$, $i = 1, \dots, n$, where R_1, \dots, R_n are the right creation operators on the full Fock space $F^2(H_n)$ and each diagonal operator G_i is defined by

$$G_i e_\alpha := \sqrt{\frac{b_\alpha}{b_{g_i \alpha}}} e_\alpha, \quad \alpha \in \mathbb{F}_n^+,$$

where the coefficients b_α , $\alpha \in \mathbb{F}_n^+$, are given by relation (1.1). In this case, we have

$$\Lambda_\beta e_\gamma = \frac{\sqrt{b_\gamma}}{\sqrt{b_{\gamma\tilde{\beta}}}} e_{\gamma\tilde{\beta}} \quad \text{and} \quad \Lambda_\beta^* e_\alpha = \begin{cases} \frac{\sqrt{b_\gamma}}{\sqrt{b_\alpha}} e_\gamma & \text{if } \alpha = \gamma\tilde{\beta} \\ 0 & \text{otherwise} \end{cases}$$

for any $\alpha, \beta \in \mathbb{F}_n^+$, where $\tilde{\beta}$ denotes the reverse of $\beta = g_{i_1} \cdots g_{i_k}$, i.e. $\tilde{\beta} := g_{i_k} \cdots g_{i_1}$. We remark that if $f := \sum_{|\alpha| \geq 1} a_\alpha Z_\alpha$ is a positive regular power series, then so is $\tilde{f} := \sum_{|\alpha| \geq 1} a_{\tilde{\alpha}} Z_\alpha$. Moreover, $\mathbf{\Lambda} := (\Lambda_1, \dots, \Lambda_n) \in \mathcal{D}_{\tilde{f}}(F^2(H_n))$ and $W_i = U^* \Lambda_i U$, where $U \in B(F^2(H_n))$ is the unitary operator defined by $U e_\alpha := e_{\tilde{\alpha}}$, $\alpha \in \mathbb{F}_n^+$. Throughout this paper, we will refer to the n -tuples $\mathbf{W} := (W_1, \dots, W_n)$ and $\mathbf{\Lambda} := (\Lambda_1, \dots, \Lambda_n)$ as the weighted creation operators associated with the regular domain \mathcal{D}_f .

In [15], we introduced the domain algebra $\mathcal{A}_n(\mathcal{D}_f)$ associated with the noncommutative domain \mathcal{D}_f to be the norm closure of all polynomials in the weighted left creation operators W_1, \dots, W_n and the identity. Using the weighted right creation operators associated with \mathcal{D}_f , one can define the corresponding domain algebra $\mathcal{R}_n(\mathcal{D}_f)$. The Hardy algebra $F_n^\infty(\mathcal{D}_f)$ (resp. $R_n^\infty(\mathcal{D}_f)$) is the w^* - (or WOT-, SOT-) closure of all polynomials in W_1, \dots, W_n (resp. $\Lambda_1, \dots, \Lambda_n$) and the identity. We proved that $F_n^\infty(\mathcal{D}_f)' = R_n^\infty(\mathcal{D}_f)$ and $R_n^\infty(\mathcal{D}_f)' = F_n^\infty(\mathcal{D}_f)$, where $'$ stands for the commutant.

Now, we recall ([13], [15]) some basic facts concerning the noncommutative Poisson kernels associated with the regular domains. Let $\mathbf{T} := (T_1, \dots, T_n)$ be an n -tuple of operators in the noncommutative

domain $\mathcal{D}_f(\mathcal{H})$, i.e. $\sum_{|\alpha| \geq 1} a_\alpha T_\alpha T_\alpha^* \leq I_{\mathcal{H}}$. Define the positive linear mapping

$$\Phi_{f,\mathbf{T}} : B(\mathcal{H}) \rightarrow B(\mathcal{H}) \quad \text{by} \quad \Phi_{f,\mathbf{T}}(X) = \sum_{|\alpha| \geq 1} a_\alpha T_\alpha X T_\alpha^*,$$

where the convergence is in the weak operator topology. We use the notation $\Phi_{f,\mathbf{T}}^m$ for the composition $\Phi_{f,\mathbf{T}} \circ \cdots \circ \Phi_{f,\mathbf{T}}$ of $\Phi_{f,\mathbf{T}}$ by itself m times. Since $\Phi_{f,\mathbf{T}}(I) \leq I$ and $\Phi_{f,\mathbf{T}}(\cdot)$ is a positive linear map, it is easy to see that $\{\Phi_{f,\mathbf{T}}^m(I)\}_{m=1}^\infty$ is a decreasing sequence of positive operators and, consequently, $Q_{f,\mathbf{T}} := \text{SOT-}\lim_{m \rightarrow \infty} \Phi_{f,\mathbf{T}}^m(I)$ exists. We say that \mathbf{T} is a *pure* n -tuple in $\mathcal{D}_f(\mathcal{H})$ if $\text{SOT-}\lim_{m \rightarrow \infty} \Phi_{f,\mathbf{T}}^m(I) = 0$. Note that, for any $\mathbf{T} := (T_1, \dots, T_n) \in \mathcal{D}_f(\mathcal{H})$ and $0 \leq r < 1$, the n -tuple $r\mathbf{T} := (rT_1, \dots, rT_n) \in \mathcal{D}_f(\mathcal{H})$ is pure. Indeed, it is enough to see that $\Phi_{f,r\mathbf{T}}^m(I) \leq r^m \Phi_{f,\mathbf{T}}^m(I) \leq r^m I$ for any $m \in \mathbb{N}$. Note also that if $\|\Phi_{f,\mathbf{T}}(I)\| < 1$, then \mathbf{T} is pure. This is due to the fact that $\|\Phi_{f,\mathbf{T}}^m(I)\| \leq \|\Phi_{f,\mathbf{T}}(I)\|^m$. We define the noncommutative Poisson kernel associated with the n -tuple $\mathbf{T} \in \mathcal{D}_f(\mathcal{H})$ to be the operator $K_{f,\mathbf{T}} : \mathcal{H} \rightarrow F^2(H_n) \otimes \mathcal{D}_{\mathbf{T}}$ defined by

$$K_{f,\mathbf{T}}h = \sum_{\alpha \in \mathbb{F}_n^+} \sqrt{b_\alpha} e_\alpha \otimes \Delta_{f,\mathbf{T}} T_\alpha^* h, \quad h \in \mathcal{H},$$

where $\Delta_{f,\mathbf{T}} := (I - \Phi_{f,\mathbf{T}}(I))^{1/2}$ is the defect operator associated with \mathbf{T} and $\mathcal{D}_{\mathbf{T}} := \overline{\Delta_{f,\mathbf{T}}(\mathcal{H})}$ is the corresponding defect space. The operator $K_{f,\mathbf{T}}$ is a contraction satisfying relation $K_{f,\mathbf{T}}^* K_{f,\mathbf{T}} = I_{\mathcal{H}} - Q_{f,\mathbf{T}}$ and

$$(1.2) \quad K_{f,\mathbf{T}} T_i^* = (W_i^* \otimes I_{\mathcal{D}_{\mathbf{T}}}) K_{f,\mathbf{T}}, \quad i = 1, \dots, n,$$

where $\mathbf{W} := (W_1, \dots, W_n)$ is the universal model associated with the noncommutative regular domain \mathcal{D}_f . Moreover, $K_{f,\mathbf{T}}$ is an isometry if and only if \mathbf{T} is pure element of $\mathcal{D}_f(\mathcal{H})$.

2. INTERTWINING DILATION THEOREM ON NONCOMMUTATIVE BI-DOMAINS

In this section, we obtain an intertwining dilation theorem which generalizes Sarason and Sz.-Nagy–Foiaş commutant lifting theorem for commuting contractions in the framework of noncommutative regular domains and Poisson kernels on weighted Fock spaces. As a consequence, we obtain a new proof for the commutant lifting theorem for pure elements in \mathcal{D}_f . More applications of this result will be considered in the next sections.

Unless otherwise specified, we assume, throughout this paper, that f and g are two positive regular polynomials in noncommutative indeterminates $\mathbf{Z} := \langle Z_1, \dots, Z_{n_1} \rangle$ and $\mathbf{Z}' := \langle Z'_1, \dots, Z'_{n_2} \rangle$, respectively, of the form

$$f := \sum_{\alpha \in \mathbb{F}_{n_1}^+, 1 \leq |\alpha| \leq k_1} a_\alpha Z_\alpha \quad \text{and} \quad g := \sum_{\beta \in \mathbb{F}_{n_2}^+, 1 \leq |\beta| \leq k_2} c_\beta Z'_\beta.$$

Fix two tuples of operators $\mathbf{T}_1 = (T_{1,1}, \dots, T_{1,n_1}) \in \mathcal{D}_f(\mathcal{H})$ and $\mathbf{T}'_1 = (T'_{1,1}, \dots, T'_{1,n_1}) \in \mathcal{D}_f(\mathcal{H}')$ and let $\mathbf{T}_2 := (T_{2,1}, \dots, T_{2,n_2})$, with $T_{2,j} : \mathcal{H}' \rightarrow \mathcal{H}$, be such that $\mathbf{T}_2 \in \mathcal{D}_g(\mathcal{H}', \mathcal{H})$ and intertwines \mathbf{T}_1 with \mathbf{T}'_1 , i.e.

$$T_{2,j} T'_{1,i} = T_{1,i} T_{2,j}$$

for any $i \in \{1, \dots, n_1\}$ and $j \in \{1, \dots, n_2\}$. We denote by $\mathcal{I}(\mathbf{T}_1, \mathbf{T}'_1)$ the set of all intertwining tuples \mathbf{T}_2 of \mathbf{T}_1 and \mathbf{T}'_1 . A straightforward calculation reveals that

$$\Delta_{f,\mathbf{T}_1}^2 + \Phi_{f,\mathbf{T}_1}(\Delta_{g,\mathbf{T}_2}^2) = \Phi_{g,\mathbf{T}_2}(\Delta_{f,\mathbf{T}'_1}^2) + \Delta_{g,\mathbf{T}_2}^2.$$

If the defect spaces $\mathcal{D}_{\mathbf{T}_1}$, $\mathcal{D}_{\mathbf{T}'_1}$, and $\mathcal{D}_{\mathbf{T}_2}$ are finite dimensional with dimensions $d_1 := \dim \mathcal{D}_{\mathbf{T}_1}$, $d'_1 := \dim \mathcal{D}_{\mathbf{T}'_1}$, and $d_2 := \dim \mathcal{D}_{\mathbf{T}_2}$, and such that $d_1 + m_1 d_2 = m_2 d'_1 + d_2$, where

$$m_i := \text{card}\{\alpha \in \mathbb{F}_{n_j}^+ : 1 \leq |\alpha| \leq k_j\}, \quad j = 1, 2,$$

then there are unitary extensions $U : \mathcal{D}_{\mathbf{T}_1} \oplus \bigoplus_{\alpha \in \mathbb{F}_{n_1}^+, 1 \leq |\alpha| \leq k_1} \mathcal{D}_{\mathbf{T}_2} \rightarrow \bigoplus_{\beta \in \mathbb{F}_{n_2}^+, 1 \leq |\beta| \leq k_2} \mathcal{D}_{\mathbf{T}'_1} \oplus \mathcal{D}_{\mathbf{T}_2}$ of the isometry

$$(2.1) \quad U \left(\Delta_{\mathbf{T}_1} h \oplus \bigoplus_{\alpha \in \mathbb{F}_{n_1}^+, 1 \leq |\alpha| \leq k_1} \sqrt{a_\alpha} \Delta_{\mathbf{T}_2} T_{1,\alpha}^* h \right) := \bigoplus_{\beta \in \mathbb{F}_{n_2}^+, 1 \leq |\beta| \leq k_2} \sqrt{c_\beta} \Delta_{\mathbf{T}'_1} T_{2,\beta}^* h, \quad h \in \mathcal{H}.$$

We denote by $\mathcal{U}_{\mathbf{T}}$ the set of all unitary extensions of the isometry given by relation (2.1). In case the above-mentioned dimensional conditions are not satisfied, then let \mathcal{K} be an infinite dimensional Hilbert space and note that the operator defined by

$$(2.2) \quad U \left(\Delta_{\mathbf{T}_1} h \oplus \bigoplus_{\alpha \in \mathbb{F}_{n_1}^+, 1 \leq |\alpha| \leq k_1} [\sqrt{a_\alpha} \Delta_{\mathbf{T}_2} T_{1,\alpha}^* h \oplus 0] \right) := \left(\bigoplus_{\beta \in \mathbb{F}_{n_2}^+, 1 \leq |\beta| \leq k_2} \sqrt{c_\beta} \Delta_{\mathbf{T}'_1} T_{2,\beta}^* h \right) \oplus 0$$

is an isometry which can be extended to a unitary operator

$$U : \mathcal{D}_{\mathbf{T}_1} \oplus \bigoplus_{\alpha \in \mathbb{F}_{n_1}^+, 1 \leq |\alpha| \leq k_1} (\mathcal{D}_{\mathbf{T}_2} \oplus \mathcal{K}) \rightarrow \bigoplus_{\beta \in \mathbb{F}_{n_2}^+, 1 \leq |\beta| \leq k_2} \mathcal{D}_{\mathbf{T}'_1} \oplus (\mathcal{D}_{\mathbf{T}_2} \oplus \mathcal{K}).$$

In this setting, we denote by $\mathcal{U}_{\mathbf{T}}^{\mathcal{K}}$ the set of all unitary extensions of the isometry defined by (2.2). Let $U = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ be the operator matrix representation of $U \in \mathcal{U}_{\mathbf{T}}^{\mathcal{K}}$, where

$$(2.3) \quad \begin{aligned} A : \mathcal{D}_{\mathbf{T}_1} &\rightarrow \bigoplus_{\beta \in \mathbb{F}_{n_2}^+, 1 \leq |\beta| \leq k_2} \mathcal{D}_{\mathbf{T}'_1}, \\ B : \bigoplus_{\alpha \in \mathbb{F}_{n_1}^+, 1 \leq |\alpha| \leq k_1} (\mathcal{D}_{\mathbf{T}_2} \oplus \mathcal{K}) &\rightarrow \bigoplus_{\beta \in \mathbb{F}_{n_2}^+, 1 \leq |\beta| \leq k_2} \mathcal{D}_{\mathbf{T}'_1}, \\ C : \mathcal{D}_{\mathbf{T}_1} &\rightarrow \mathcal{D}_{\mathbf{T}_2} \oplus \mathcal{K}, \text{ and} \\ D : \bigoplus_{\alpha \in \mathbb{F}_{n_1}^+, 1 \leq |\alpha| \leq k_1} (\mathcal{D}_{\mathbf{T}_2} \oplus \mathcal{K}) &\rightarrow \mathcal{D}_{\mathbf{T}_2} \oplus \mathcal{K}. \end{aligned}$$

Given an operator $Z : \mathcal{N} \rightarrow \mathcal{M}$ and $n \in \mathbb{N}$, we introduce the ampliation

$$\text{diag}_n(Z) := \begin{pmatrix} Z & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & Z \end{pmatrix} : \bigoplus_{s=1}^n \mathcal{N} \rightarrow \bigoplus_{s=1}^n \mathcal{M}.$$

In what follows, we consider the lexicographic order for the free semigroup $\mathbb{F}_{n_1}^+$, that is

$$g_0 < g_1 < \cdots < g_{n_1} < g_1 g_1 < \cdots < g_1 g_{n_1} < \cdots < g_{n_1} g_1 < \cdots < g_{n_1} g_{n_1} < \cdots$$

and so on. For the direct product $\mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_1}^+$ of p copies of $\mathbb{F}_{n_1}^+$, we say that $(\alpha_p, \dots, \alpha_1) < (\beta_p, \dots, \beta_1)$ if $\alpha_p < \beta_p$ or there is $i \in \{2, \dots, p\}$ such that $\alpha_p = \beta_p, \dots, \alpha_i = \beta_i$, and $\alpha_{i-1} < \beta_{i-1}$. We also use the

operator column notation $\begin{bmatrix} Y_{(\alpha_p, \dots, \alpha_1)} \\ \vdots \\ \alpha_i \in \mathbb{F}_{n_1}^+, 1 \leq |\alpha_i| \leq k \end{bmatrix}$, where the entries $Y_{(\alpha_p, \dots, \alpha_1)}$ are arranged in the order

mentioned above. For simplicity, $[X_1, \dots, X_n]$ denotes either the n -tuple $(X_1, \dots, X_n) \in B(\mathcal{H})^n$ or the operator row matrix $[X_1 \cdots X_n]$ acting from $\mathcal{H}^{(n)}$, the direct sum of n copies of the Hilbert space \mathcal{H} , to \mathcal{H} .

Lemma 2.1. *If $\mathbf{X}_1 := [\sqrt{a_\alpha}T_{1,\alpha} : \alpha \in \mathbb{F}_{n_1}^+, 1 \leq |\alpha| \leq k_1]$ and $m_1 := \text{card}\{\alpha \in \mathbb{F}_{n_1}^+ : 1 \leq |\alpha| \leq k_1\}$, then*

$$\begin{aligned} & \text{diag}_{m_1} \left(D \text{diag}_{m_1} \left(\cdots D \text{diag}_{m_1} \left(\widehat{\Delta}_{\mathbf{T}_2} \right) \mathbf{X}_1^* \cdots \right) \mathbf{X}_1^* \right) \mathbf{X}_1^* \\ &= \text{diag}_{m_1} \left(D \text{diag}_{m_1} \left(\cdots D \text{diag}_{m_1} \left(\widehat{\Delta}_{\mathbf{T}_2} \right) \cdots \right) \right) \begin{bmatrix} \sqrt{a_{\alpha_1} \cdots a_{\alpha_p}} (T_{1,\alpha_p} \cdots T_{1,\alpha_1})^* \\ \vdots \\ \alpha_i \in \mathbb{F}_{n_1}^+, 1 \leq |\alpha_i| \leq k_1 \end{bmatrix}, \end{aligned}$$

where diag_{m_1} appears p times on each side of the equality, and $\widehat{\Delta}_{\mathbf{T}_2} := \begin{bmatrix} \Delta_{\mathbf{T}_2} \\ 0 \end{bmatrix} : \mathcal{H} \rightarrow \mathcal{D}_{\mathbf{T}_2} \oplus \mathcal{K}$.

Proof. Let $D = [D_{(\alpha)} : \alpha \in \mathbb{F}_{n_1}^+, 1 \leq |\alpha| \leq k_1]$ with $D_{(\alpha)} \in B(\mathcal{D}_{\mathbf{T}_2} \oplus \mathcal{K})$ and note that

$$D \text{diag}_{m_1} \left(\widehat{\Delta}_{\mathbf{T}_2} \right) \mathbf{X}_1^* = \sum_{\alpha_1 \in \mathbb{F}_{n_1}^+, 1 \leq |\alpha_1| \leq k_1} D_{(\alpha_1)} \widehat{\Delta}_{\mathbf{T}_2} \sqrt{a_{\alpha_1}} T_{1,\alpha_1}^*$$

and

$$\text{diag}_{m_1} \left(D \text{diag}_{m_1} \left(\widehat{\Delta}_{\mathbf{T}_2} \right) \mathbf{X}_1^* \right) \mathbf{X}_1^* = \begin{bmatrix} \sum_{\substack{\alpha_1 \in \mathbb{F}_{n_1}^+ \\ 1 \leq |\alpha_1| \leq k_1}} D_{(\alpha_1)} \widehat{\Delta}_{\mathbf{T}_2} \sqrt{a_{\alpha_1}} \sqrt{a_{\alpha_2}} (T_{1,\alpha_2} T_{1,\alpha_1})^* \\ \vdots \\ \alpha_2 \in \mathbb{F}_{n_1}^+, 1 \leq |\alpha_2| \leq k_1 \end{bmatrix}.$$

An inductive argument shows that

$$\begin{aligned} & \text{diag}_{m_1} \left(D \text{diag}_{m_1} \left(\cdots D \text{diag}_{m_1} \left(\widehat{\Delta}_{\mathbf{T}_2} \right) \mathbf{X}_1^* \cdots \right) \mathbf{X}_1^* \right) \mathbf{X}_1^* \\ &= \begin{bmatrix} \sum_{\substack{\alpha_1, \dots, \alpha_{p-1} \in \mathbb{F}_{n_1}^+ \\ 1 \leq |\alpha_i| \leq k_1}} D_{(\alpha_{p-1})} \cdots D_{(\alpha_1)} \widehat{\Delta}_{\mathbf{T}_2} \sqrt{a_{\alpha_1} \cdots a_{\alpha_{p-1}} a_{\alpha_p}} (T_{1,\alpha_p} \alpha_{p-1} \cdots \alpha_1)^* \\ \vdots \\ \alpha_p \in \mathbb{F}_{n_1}^+, 1 \leq |\alpha_p| \leq k_1 \end{bmatrix}, \end{aligned}$$

where diag_{m_1} appears p times. On the other hand, one can easily prove by induction that

$$\begin{aligned} & \text{diag}_{m_1} \left(D \text{diag}_{m_1} \left(\cdots D \text{diag}_{m_1} \left(\widehat{\Delta}_{\mathbf{T}_2} \right) \cdots \right) \right) \begin{bmatrix} \sqrt{a_{\alpha_1} \cdots a_{\alpha_p}} (T_{1,\alpha_p} \cdots T_{1,\alpha_1})^* \\ \vdots \\ \alpha_i \in \mathbb{F}_{n_1}^+, 1 \leq |\alpha_i| \leq p \end{bmatrix} \\ &= \text{diag}_{m_1} \left([D_{(\alpha_{p-1})} \cdots D_{(\alpha_1)} \widehat{\Delta}_{\mathbf{T}_2} : \alpha_1, \dots, \alpha_{p-1} \in \mathbb{F}_{n_1}^+, 1 \leq |\alpha_i| \leq k_1] \right) \\ & \quad \times \begin{bmatrix} \begin{bmatrix} \sqrt{a_{\alpha_1} \cdots a_{\alpha_{p-1}} a_{\alpha_p}} (T_{1,\alpha_p} \alpha_{p-1} \cdots \alpha_1)^* \\ \vdots \\ \alpha_1, \dots, \alpha_{p-1} \in \mathbb{F}_{n_1}^+, 1 \leq |\alpha_i| \leq k_1 \end{bmatrix} \\ \vdots \\ \alpha_p \in \mathbb{F}_{n_1}^+, 1 \leq |\alpha_p| \leq k_1 \end{bmatrix}. \end{aligned}$$

The proof is complete. \square

Lemma 2.2. *Let $\mathbf{T}_2 \in \mathcal{I}(\mathbf{T}_1, \mathbf{T}'_1)$ and let $U = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ be the matrix representation of a unitary extension $U \in \mathcal{U}_{\mathbf{T}}^{\mathcal{K}}$ (see relation (2.3)). If*

$$\begin{aligned} \mathbf{X}_1 &:= [\sqrt{a_\alpha}T_{1,\alpha} : \alpha \in \mathbb{F}_{n_1}^+, 1 \leq |\alpha| \leq k_1], \\ m_j &:= \text{card}\{\alpha \in \mathbb{F}_{n_j}^+ : 1 \leq |\alpha| \leq k_j\}, \quad j = 1, 2, \end{aligned}$$

and \mathbf{T}_1 is a pure element in $\mathcal{D}_f(\mathcal{H})$, then

$$\begin{aligned} \text{diag}_{m_2}(\Delta_{\mathbf{T}'_1}) \begin{bmatrix} \sqrt{c_\beta} T_{2,\beta}^* \\ \vdots \\ \beta \in \mathbb{F}_{n_2}^+, 1 \leq |\beta| \leq k_2 \end{bmatrix} \\ = A\Delta_{\mathbf{T}_1}h + B \sum_{p=0}^{\infty} \text{diag}_{m_1} (D\text{diag}_{m_1} (\cdots D\text{diag}_{m_1} (C\Delta_{\mathbf{T}_1}) \mathbf{X}_1^* \cdots) \mathbf{X}_1^*) \mathbf{X}_1^* h, \end{aligned}$$

for any $h \in \mathcal{H}$, where diag_{m_1} appears $p+1$ times in the general term of the series.

Proof. Due to relation (2.2), we have

$$(2.4) \quad A\Delta_{\mathbf{T}_1}h + B \begin{bmatrix} \begin{bmatrix} \Delta_{\mathbf{T}_2} \sqrt{a_\alpha} T_{1,\alpha}^* h \\ 0 \\ \vdots \\ \alpha \in \mathbb{F}_{n_1}^+, 1 \leq |\alpha| \leq k_1 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} \Delta_{\mathbf{T}'_1} \sqrt{c_\beta} T_{2,\beta}^* \\ \vdots \\ \beta \in \mathbb{F}_{n_2}^+, 1 \leq |\beta| \leq k_2 \end{bmatrix}$$

and

$$(2.5) \quad C\Delta_{\mathbf{T}_1}h + D \begin{bmatrix} \begin{bmatrix} \Delta_{\mathbf{T}_2} \sqrt{a_\alpha} T_{1,\alpha}^* h \\ 0 \\ \vdots \\ \alpha \in \mathbb{F}_{n_1}^+, 1 \leq |\alpha| \leq k_1 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} \Delta_{\mathbf{T}_2} h \\ 0 \end{bmatrix}$$

for any $h \in \mathcal{H}$. Since $\hat{\Delta}_{\mathbf{T}_2} := \begin{bmatrix} \Delta_{\mathbf{T}_2} \\ 0 \end{bmatrix} : \mathcal{H} \rightarrow \mathcal{D}_{\mathbf{T}_2} \oplus \mathcal{K}$, we can rewrite relations (2.4) and (2.5) as

$$(2.6) \quad A\Delta_{\mathbf{T}_1} + B\text{diag}_{m_1}(\hat{\Delta}_{\mathbf{T}_2})\mathbf{X}_1^* = \text{diag}_{m_2}(\Delta_{\mathbf{T}'_1})\mathbf{X}_2^*,$$

where $\mathbf{X}_2 := [\sqrt{c_\beta} T_{2,\beta} : \beta \in \mathbb{F}_{n_2}^+, 1 \leq |\beta| \leq k_2]$, and

$$(2.7) \quad C\Delta_{\mathbf{T}_1} + D\text{diag}_{m_1}(\hat{\Delta}_{\mathbf{T}_2})\mathbf{X}_1^* = \hat{\Delta}_{\mathbf{T}_2},$$

respectively. Note that using relation (2.7) we deduce that

$$(2.8) \quad \text{diag}_{m_1}(\hat{\Delta}_{\mathbf{T}_2})\mathbf{X}_1^* = \text{diag}_{m_1}(C\Delta_{\mathbf{T}_1})\mathbf{X}_1^* + \text{diag}_{m_1}(D\text{diag}_{m_1}(\hat{\Delta}_{\mathbf{T}_2})\mathbf{X}_1^*)\mathbf{X}_1^*,$$

which combined with relation (2.6) yields

$$\text{diag}_{m_2}(\Delta_{\mathbf{T}'_1})\mathbf{X}_2^* = A\Delta_{\mathbf{T}_1} + B\text{diag}_{m_1}(C\Delta_{\mathbf{T}_1})\mathbf{X}_1^* + B\text{diag}_{m_1}(D\text{diag}_{m_1}(\hat{\Delta}_{\mathbf{T}_2})\mathbf{X}_1^*)\mathbf{X}_1^*.$$

Continuing to use relation (2.8) in the latter relation and the resulting ones, an induction argument leads to the identity

$$(2.9) \quad \begin{aligned} \text{diag}_{m_2}(\Delta_{\mathbf{T}'_1})\mathbf{X}_2^* &= A\Delta_{\mathbf{T}_1} + B\text{diag}_{m_1}(C\Delta_{\mathbf{T}_1})\mathbf{X}_1^* \\ &+ B \sum_{p=1}^m \text{diag}_{m_1} (D\text{diag}_{m_1} (\cdots D\text{diag}_{m_1} (C\Delta_{\mathbf{T}_1}) \mathbf{X}_1^* \cdots) \mathbf{X}_1^*) \mathbf{X}_1^* \\ &+ B\text{diag}_{m_1} (D\text{diag}_{m_1} (\cdots D\text{diag}_{m_1} (\hat{\Delta}_{\mathbf{T}_2}) \mathbf{X}_1^* \cdots) \mathbf{X}_1^*) \mathbf{X}_1^*, \end{aligned}$$

where diag_{m_1} appears $p+1$ times in the general term of the sum above and $m+2$ times in the last term. Since $\Delta_{\mathbf{T}_2}$ and D are contractions and due to Lemma 2.1, one can easily see that

$$\begin{aligned} &\|B\text{diag}_{m_1} (D\text{diag}_{m_1} (\cdots D\text{diag}_{m_1} (\hat{\Delta}_{\mathbf{T}_2}) \mathbf{T}_1^* \cdots) \mathbf{T}_1^*) \mathbf{T}_1^* h\| \\ &\leq \|B\| \left(\sum_{\substack{\alpha_1, \dots, \alpha_{m+2} \in \mathbb{F}_{n_1}^+ \\ 1 \leq |\alpha_i| \leq k_1}} \|\sqrt{a_{\alpha_1} \cdots a_{\alpha_{m+2}}} (T_{1,\alpha_{m+2}} \cdots T_{1,\alpha_1})^* h\|^2 \right)^{1/2} = \langle \Phi_{f, \mathbf{T}_1}^{m+2}(I)h, h \rangle \end{aligned}$$

for any $h \in \mathcal{H}$. Since \mathbf{T}_1 is pure in $\mathcal{D}_f(\mathcal{H})$, we have $\lim_{m \rightarrow \infty} \Phi_{f, \mathbf{T}_1}^{m+2}(I)h = 0$ for any $h \in \mathcal{H}$. Consequently, relation (2.9) implies

$$\text{diag}_{m_2}(\Delta_{\mathbf{T}'_1})\mathbf{X}_2^*h = A\Delta_{\mathbf{T}_1}h + B \sum_{p=0}^{\infty} \text{diag}_{m_1} (D \text{diag}_{m_1} (\cdots D \text{diag}_{m_1} (C\Delta_{\mathbf{T}_1}) \mathbf{X}_1^* \cdots) \mathbf{X}_1^*) \mathbf{X}_1^*h,$$

for any $h \in \mathcal{H}$, where diag_{m_1} appears $p+1$ times in the general term of the series. The proof is complete. \square

We recall ([12], [15]) a few facts concerning multi-analytic operators on Fock spaces. We say that a bounded linear operator M acting from $F^2(H_n) \otimes \mathcal{K}$ to $F^2(H_n) \otimes \mathcal{K}'$ is multi-analytic with respect to the universal model $\mathbf{W} := (W_1, \dots, W_n)$ if $M(W_i \otimes I_{\mathcal{K}}) = (W_i \otimes I_{\mathcal{K}'})M$ for any $i = 1, \dots, n$. We can associate with M a unique formal Fourier expansion $\sum_{\alpha \in \mathbb{F}_n^+} \Lambda_{\alpha} \otimes \theta_{(\alpha)}$ where $\theta_{(\alpha)} \in B(\mathcal{K}, \mathcal{K}')$. We know that $M = \text{SOT-lim}_{r \rightarrow 1} \sum_{k=0}^{\infty} \sum_{|\alpha|=k} r^{|\alpha|} \Lambda_{\alpha} \otimes \theta_{(\alpha)}$ where, for each $r \in [0, 1)$, the series converges in the uniform norm. Moreover, the set of all multi-analytic operators in $B(F^2(H_n) \otimes \mathcal{K}, F^2(H_n) \otimes \mathcal{K}')$ coincides with $\mathcal{R}_n^{\infty}(\mathcal{D}_f) \bar{\otimes} B(\mathcal{K}, \mathcal{K}')$, the WOT-closed operator space generated by the spatial tensor product.

Let \mathcal{H} , \mathcal{H}' , and \mathcal{E} be Hilbert spaces and consider

$$U = \begin{bmatrix} A & B \\ C & D \end{bmatrix} : \begin{array}{c} \mathcal{H} \\ \oplus \\ \bigoplus_{\alpha \in \mathbb{F}_{n_1}^+, 1 \leq |\alpha| \leq k_1} \mathcal{E} \end{array} \xrightarrow{\beta \in \mathbb{F}_{n_2}^+, 1 \leq |\beta| \leq k_2} \begin{array}{c} \mathcal{H}' \\ \oplus \\ \bigoplus_{\alpha \in \mathbb{F}_{n_1}^+, 1 \leq |\alpha| \leq k_1} \mathcal{E} \end{array}$$

to be a unitary operator. Setting $D = [D_{(\alpha)} : \alpha \in \mathbb{F}_{n_1}^+, 1 \leq |\alpha| \leq 1] : \bigoplus_{\alpha \in \mathbb{F}_{n_1}^+, 1 \leq |\alpha| \leq k_1} \mathcal{E} \rightarrow \mathcal{E}$, we associate

with U^* and any $r \in [0, 1)$ the operator $\varphi_{U^*}(r\mathbf{\Lambda}_1)$ defined by

$$\begin{aligned} \varphi_{U^*}(r\mathbf{\Lambda}_1) &:= I_{F^2(H_{n_1})} \otimes A^* + \left(I_{F^2(H_{n_1})} \otimes C^* \right) \left(I_{F^2(H_{n_1}) \otimes \mathcal{E}} - \sum_{\substack{\alpha \in \mathbb{F}_{n_1}^+ \\ 1 \leq |\alpha| \leq k_1}} r^{|\alpha|} \sqrt{a_{\alpha}} \Lambda_{1, \tilde{\alpha}} \otimes D_{(\alpha)}^* \right)^{-1} \\ &\quad \times \left[\sqrt{a_{\alpha}} \Lambda_{1, \tilde{\alpha}} \otimes I_{\mathcal{H}} : \alpha \in \mathbb{F}_{n_1}^+, 1 \leq |\alpha| \leq k_1 \right] \left(I_{F^2(H_{n_1})} \otimes B^* \right), \end{aligned}$$

where $\mathbf{\Lambda}_1 := [\Lambda_{1,1}, \dots, \Lambda_{1,n_1}]$ is the tuple of weighted right creation operators on $F^2(H_{n_1})$ associated with the regular domain \mathcal{D}_f . In what follows, we use the notations: $\mathbf{A} := I_{F^2(H_{n_1})} \otimes A$, $\mathbf{B} := I_{F^2(H_{n_1})} \otimes B$, $\mathbf{C} := I_{F^2(H_{n_1})} \otimes C$, $\mathbf{D} := I_{F^2(H_{n_1})} \otimes D$, and $\mathbf{\Gamma}(r) := [\sqrt{a_{\alpha}} r^{|\alpha|} \Lambda_{1, \tilde{\alpha}} \otimes I_{\mathcal{E}} : \alpha \in \mathbb{F}_{n_1}^+, 1 \leq |\alpha| \leq k_1]$.

Lemma 2.3. *The strong operator topology limit $\varphi_{U^*}(\mathbf{\Lambda}_1) := \text{SOT-lim}_{r \rightarrow 1} \varphi_{U^*}(r\mathbf{\Lambda}_1)$ exists and defines a contractive multi-analytic operator with respect to \mathbf{W}_1 having the row matrix representation*

$$\varphi_{U^*}(\mathbf{\Lambda}_1) = [\varphi_{(\beta)}(\mathbf{\Lambda}_1) : \beta \in \mathbb{F}_{n_2}^+, 1 \leq |\beta| \leq k_2],$$

with $\varphi_{(\beta)}(\mathbf{\Lambda}_1) \in \mathcal{R}_{n_1}^{\infty}(\mathcal{D}_f) \bar{\otimes} B(\mathcal{H}', \mathcal{H})$, where $\mathcal{R}_{n_1}^{\infty}(\mathcal{D}_f)$ is the noncommutative Hardy algebra generated by the weighted right creation operators $\Lambda_{1,1}, \dots, \Lambda_{1,n_1}$ and the identity.

Proof. Since \mathbf{D} and $\mathbf{\Gamma}(r)$ are contractions, we have $\left\| \sum_{\alpha \in \mathbb{F}_{n_1}^+, 1 \leq |\alpha| \leq k_1} r^{|\alpha|} \sqrt{a_{\alpha}} \Lambda_{1, \tilde{\alpha}} \otimes D_{(\alpha)}^* \right\| \leq r < 1$. Con-

sequently, the operator $\varphi_{U^*}(r\mathbf{\Lambda}_1)$ makes sense and $\varphi_{U^*}(r\mathbf{\Lambda}_1) = [\varphi_{(\beta)}(r\mathbf{\Lambda}_1) : \beta \in \mathbb{F}_{n_2}^+, 1 \leq |\beta| \leq k_2]$ with $\varphi_{(\beta)}(r\mathbf{\Lambda}_1) \in \mathcal{R}_{n_1}(\mathcal{D}_f) \bar{\otimes} B(\mathcal{H}', \mathcal{H})$, where $\mathcal{R}_{n_1}(\mathcal{D}_f)$ is the noncommutative disk algebra generated by $\Lambda_{1,1}, \dots, \Lambda_{1,n_1}$ and the identity. Consequently,

$$(2.10) \quad \varphi_{(\beta)}(r\mathbf{\Lambda}_1) = \sum_{k=0}^{\infty} \sum_{\gamma \in \mathbb{F}_{n_1}^+, |\gamma|=k} r^{|\gamma|} \mathbf{\Lambda}_{1,\gamma} \otimes \Theta_{(\gamma)}^{(\beta)}$$

for some operators $\Theta_{(\gamma)}^{(\beta)} \in B(\mathcal{H}', \mathcal{H})$, where the convergence is in the operator norm topology. On the other hand, since $\begin{bmatrix} \mathbf{A}^* & \mathbf{C}^* \\ \mathbf{B}^* & \mathbf{D}^* \end{bmatrix}$ is a unitary operator, standard calculations (see e.g. [14]) show that

$$\begin{aligned} I - \varphi_{U^*}(r\mathbf{A}_1)\varphi_{U^*}(r\mathbf{A}_1)^* \\ = \mathbf{C}^*(I - \mathbf{\Gamma}(r)\mathbf{D}^*)^{-1} \left[\left(I - \sum_{\alpha \in \mathbb{F}_{n_1}^+, 1 \leq |\alpha| \leq k_1} r^{2|\alpha|} a_\alpha \Lambda_{1,\tilde{\alpha}} \Lambda_{1,\tilde{\alpha}}^* \right) \otimes I \right] (I - \mathbf{D}\mathbf{\Gamma}(r)^*)^{-1} \mathbf{C}. \end{aligned}$$

This shows that $\varphi_{U^*}(r\mathbf{A}_1)$ is a contraction for any $r \in [0, 1)$ having the row matrix representation $\varphi_{U^*}(r\mathbf{A}_1) = [\varphi_{(\beta)}(r\mathbf{A}_1) : \beta \in \mathbb{F}_{n_2}^+, 1 \leq |\beta| \leq k_2]$ with $\varphi_{(\beta)}(r\mathbf{A}_1) \in \mathcal{R}_{n_1}(\mathcal{D}_f) \otimes B(\mathcal{H}', \mathcal{H})$. Since $\varphi_{(\beta)}(r\mathbf{A}_1)$ has the Fourier representation (2.10), one can see that $\varphi_{(\beta)}(\mathbf{A}_1) := \text{SOT-lim}_{r \rightarrow 1} \varphi_{(\beta)}(r\mathbf{A}_1)$ exists, $\varphi_{(\beta)}(\mathbf{A}_1) \in \mathcal{R}_{n_1} \otimes B(\mathcal{H}', \mathcal{H})$, and $\|\varphi_{U^*}(\mathbf{A}_1)\| \leq 1$. The proof is complete. \square

We recall that $\mathcal{I}(\mathbf{T}_1, \mathbf{T}'_1)$ is the set of all tuples $\mathbf{T}_2 := (T_{2,1}, \dots, T_{2,n_2})$, with $T_{2,j} : \mathcal{H}' \rightarrow \mathcal{H}$, such that $\mathbf{T}_2 \in \mathcal{D}_g(\mathcal{H}', \mathcal{H})$ intertwines \mathbf{T}_1 with \mathbf{T}'_1 , i.e. $T_{2,j}T'_{1,i} = T_{1,i}T_{2,j}$ for any $i \in \{1, \dots, n_1\}$ and $j \in \{1, \dots, n_2\}$. The main result of this section is the following intertwining dilation theorem for the elements of $\mathcal{I}(\mathbf{T}_1, \mathbf{T}'_1)$.

Theorem 2.4. *Let $\mathbf{T}_1 := (T_{1,1}, \dots, T_{1,n_1}) \in \mathcal{D}_f(\mathcal{H})$ and $\mathbf{T}'_1 := (T'_{1,1}, \dots, T'_{1,n_1}) \in \mathcal{D}_f(\mathcal{H}')$, and let $\mathbf{T}_2 := (T_{2,1}, \dots, T_{2,n_2}) \in \mathcal{D}_g(\mathcal{H}', \mathcal{H})$ be such that $\mathbf{T}_2 \in \mathcal{I}(\mathbf{T}_1, \mathbf{T}'_1)$. Let $\mathbf{W}_1 := (W_{1,1}, \dots, W_{1,n_1})$ and $\mathbf{A}_1 := (\Lambda_{1,1}, \dots, \Lambda_{1,n_1})$ be the weighted creation operators associated with the noncommutative domain \mathcal{D}_f . If $\varphi_{U^*}(\mathbf{A}_1) = [\varphi_{(\beta)}(\mathbf{A}_1) : \beta \in \mathbb{F}_{n_2}^+, 1 \leq |\beta| \leq k_2]$ is the contractive multi-analytic operator associated with $U \in \mathcal{U}_{\mathbf{T}}^{\mathcal{K}}$ and \mathbf{T}_1 is a pure element of the noncommutative regular domain $\mathcal{D}_f(\mathcal{H})$, then the following relations hold:*

$$K_{f, \mathbf{T}'_1} T_{2, \beta}^* = \frac{1}{\sqrt{c_\beta}} \varphi_{(\beta)}(\mathbf{A}_1)^* K_{f, \mathbf{T}_1}, \quad \beta \in \mathbb{F}_{n_2}^+, 1 \leq |\beta| \leq k_2,$$

and

$$K_{f, \mathbf{T}_1} T_{1, i}^* = (W_{1, i}^* \otimes I_{\mathcal{D}_{\mathbf{T}_1}}) K_{f, \mathbf{T}_1}, \quad K_{f, \mathbf{T}'_1} (T'_{1, i})^* = (W_{1, i}^* \otimes I_{\mathcal{D}_{\mathbf{T}'_1}}) K_{f, \mathbf{T}'_1}, \quad i \in \{1, \dots, n_1\},$$

where K_{f, \mathbf{T}_1} and K_{f, \mathbf{T}'_1} are the noncommutative Poisson kernels associated with \mathbf{T}_1 and \mathbf{T}'_1 , respectively.

Proof. Fix $U = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathcal{U}_{\mathbf{T}}^{\mathcal{K}}$ and set

$$D := [D_{(\alpha)} : \alpha \in \mathbb{F}_{n_1}^+, 1 \leq |\alpha| \leq 1] : \bigoplus_{\alpha \in \mathbb{F}_{n_1}^+, 1 \leq |\alpha| \leq k_1} (\mathcal{D}_{\mathbf{T}_2} \oplus \mathcal{K}) \rightarrow \mathcal{D}_{\mathbf{T}_2} \oplus \mathcal{K}.$$

In what follows, we use the notations: $\mathbf{A} := I_{F^2(H_{n_1})} \otimes A$, $\mathbf{B} := I_{F^2(H_{n_1})} \otimes B$, $\mathbf{C} := I_{F^2(H_{n_1})} \otimes C$,

$$\mathbf{Q} := \sum_{\alpha \in \mathbb{F}_{n_1}^+, 1 \leq |\alpha| \leq k_1} a_\alpha \Lambda_{1, \tilde{\alpha}} \otimes D_{(\alpha)}^* \text{ and } \mathbf{\Gamma} := [\sqrt{a_\sigma} \Lambda_{1, \tilde{\sigma}} \otimes I_{\mathcal{D}_{\mathbf{T}_2} \oplus \mathcal{K}} : \sigma \in \mathbb{F}_{n_1}^+, 1 \leq |\sigma| \leq k_1].$$

As in Lemma 2.1, an induction argument over q shows that

$$(2.11) \quad \text{diag}_{m_1} (\text{diag}_{m_1} \cdots (\text{diag}_{m_1} (C^*) D^*) \cdots D^*) = \text{diag}_{m_1} \left(\begin{bmatrix} C^* (D_{(\gamma_1)} \cdots D_{(\gamma_q)})^* \\ \vdots \\ \gamma_i \in \mathbb{F}_{n_1}^+, 1 \leq |\gamma_i| \leq k_1 \end{bmatrix} \right),$$

where diag_{m_1} appears $q + 1$ times on the left-hand side of the equality.

We associate with U^* the multi-analytic operator $\varphi_{U^*}(\mathbf{A}_1) \in \mathcal{R}_{n_1}^\infty(\mathcal{D}_f) \otimes B(\bigoplus_{\beta \in \mathbb{F}_{n_2}^+, 1 \leq |\beta| \leq k_2} \mathcal{D}_{\mathbf{T}'_1}, \mathcal{D}_{\mathbf{T}_1})$, as in Lemma 2.3, in the particular case when $\mathcal{H} = \mathcal{D}_{\mathbf{T}_1}$, $\mathcal{H}' = \mathcal{D}_{\mathbf{T}'_1}$, and $\mathcal{E} = \mathcal{D}_{\mathbf{T}_2} \oplus \mathcal{K}$. Note that, for each $y \in \bigoplus_{\beta \in \mathbb{F}_{n_2}^+, 1 \leq |\beta| \leq k_2} \mathcal{D}_{\mathbf{T}'_1}$ and $\alpha \in \mathbb{F}_{n_1}^+$ with $|\alpha| = n$, we have

$$\varphi_{U^*}(\mathbf{A}_1)(e_\alpha \otimes y) = e_\alpha \otimes A^* y + \sum_{q=0}^{\infty} \mathbf{C}^* \mathbf{Q}^q \mathbf{\Gamma} \mathbf{B}^*(e_\alpha \otimes y)$$

and $\mathbf{C}^*\mathbf{Q}^q\mathbf{TB}^*(e_\alpha \otimes y)$ is in the closed linear span of all the vectors $e_{\alpha\alpha_1\cdots\alpha_{q+1}} \otimes z$, where $\alpha_1, \dots, \alpha_{q+1} \in \mathbb{F}_{n_1}^+$, $1 \leq |\alpha_i| \leq k_1$ and $z \in \mathcal{D}_{\mathbf{T}_1}$. Consequently, using the noncommutative Poisson kernel $K_{\mathbf{T}_1}$, we deduce that

$$\begin{aligned}
 & \langle \varphi_{U^*}(\mathbf{\Lambda}_1)^* K_{f, \mathbf{T}_1} h, e_\alpha \otimes y \rangle \\
 &= \left\langle \sum_{k=0}^{\infty} \sum_{\beta \in \mathbb{F}_{n_1}^+, |\beta|=k} \sqrt{b_\beta} e_\beta \otimes \Delta_{\mathbf{T}_1} T_{1,\beta}^* h, \varphi_{U^*}(\mathbf{\Lambda}_1)(e_\alpha \otimes y) \right\rangle \\
 &= \left\langle \sqrt{b_\alpha} \Delta_{\mathbf{T}_1} T_{1,\alpha}^* h, A^* y \right\rangle + \sum_{q=0}^{\infty} \left\langle \sum_{\gamma \in \mathbb{F}_{n_1}^+} \sqrt{b_{\alpha\gamma}} e_{\alpha\gamma} \otimes \Delta_{\mathbf{T}_1} T_{1,\gamma}^* T_{1,\alpha}^* h, \mathbf{C}^*\mathbf{Q}^q\mathbf{TB}^*(e_\alpha \otimes y) \right\rangle,
 \end{aligned} \tag{2.12}$$

for any $h \in \mathcal{H}$, $y \in \bigoplus_{\beta \in \mathbb{F}_{n_2}^+, 1 \leq |\beta| \leq k_2} \mathcal{D}_{\mathbf{T}'_1}$, and $\alpha \in \mathbb{F}_{n_1}^+$ with $|\alpha| = n$. Setting

$$B := [B_{(\alpha)} : \alpha \in \mathbb{F}_{n_1}^+, 1 \leq |\alpha| \leq 1] : \bigoplus_{\alpha \in \mathbb{F}_{n_1}^+, 1 \leq |\alpha| \leq k_1} (\mathcal{D}_{\mathbf{T}_2} \oplus \mathcal{K}) \rightarrow \bigoplus_{\beta \in \mathbb{F}_{n_2}^+, 1 \leq |\beta| \leq k_2} \mathcal{D}_{\mathbf{T}'_1},$$

we obtain

$$\begin{aligned}
 & \mathbf{C}^*\mathbf{Q}^q\mathbf{TB}^*(e_\alpha \otimes y) \\
 &= (I_{F^2(H_{n_1})} \otimes C^*) \left(\sum_{\substack{\alpha_1, \dots, \alpha_q \in \mathbb{F}_{n_1}^+ \\ 1 \leq |\alpha_i| \leq k_1}} \sqrt{a_{\alpha_1} \cdots a_{\alpha_q}} \Lambda_{1, \tilde{\alpha}_1} \cdots \Lambda_{1, \tilde{\alpha}_q} \otimes D_{(\alpha_1)}^* \cdots D_{(\alpha_q)}^* \right) \\
 & \quad \left[\sqrt{a_\sigma} \Lambda_{1, \tilde{\sigma}} \otimes I_{\mathcal{D}_{\mathbf{T}_2} \oplus \mathcal{K}} : \sigma \in \mathbb{F}_{n_1}^+, 1 \leq |\sigma| \leq k_1 \right] \begin{bmatrix} I_{F^2(H_{n_1})} \otimes B_{(\sigma)}^* \\ \vdots \\ \sigma \in \mathbb{F}_{n_1}^+, 1 \leq |\sigma| \leq k_1 \end{bmatrix} (e_\alpha \otimes y) \\
 &= \sum_{\substack{\sigma \in \mathbb{F}_{n_1}^+ \\ 1 \leq |\sigma| \leq k_1}} \left(\sum_{\substack{\alpha_1, \dots, \alpha_q \in \mathbb{F}_{n_1}^+ \\ 1 \leq |\alpha_i| \leq k_1}} \sqrt{a_{\alpha_1} \cdots a_{\alpha_q}} a_\sigma \Lambda_{1, \tilde{\alpha}_1} \cdots \Lambda_{1, \tilde{\alpha}_q} \Lambda_{1, \tilde{\sigma}} e_\alpha \otimes C^* D_{(\alpha_1)}^* \cdots D_{(\alpha_q)}^* B_{(\sigma)}^* y \right) \\
 &= \sum_{\substack{\sigma \in \mathbb{F}_{n_1}^+ \\ 1 \leq |\sigma| \leq k_1}} \left(\sum_{\substack{\alpha_1, \dots, \alpha_q \in \mathbb{F}_{n_1}^+ \\ 1 \leq |\alpha_i| \leq k_1}} \sqrt{a_{\alpha_1} \cdots a_{\alpha_q}} a_\sigma \frac{\sqrt{b_\alpha}}{\sqrt{b_{\alpha\sigma\alpha_q \cdots \alpha_1}}} e_{\alpha\sigma\alpha_q \cdots \alpha_1} C^* D_{(\alpha_1)}^* \cdots D_{(\alpha_q)}^* B_{(\sigma)}^* y \right),
 \end{aligned}$$

where $\tilde{\sigma}$ is the reverse of $\sigma \in \mathbb{F}_{n_1}^+$. Consequently, using relation (2.11), we deduce that

$$\begin{aligned}
& \sum_{q=0}^{\infty} \left\langle \sum_{\gamma \in \mathbb{F}_{n_1}^+} \sqrt{b_{\alpha\gamma}} e_{\alpha\gamma} \otimes \Delta_{\mathbf{T}_1} T_{1,\gamma}^* T_{1,\alpha}^* h, \mathbf{C}^* \mathbf{Q}^q \mathbf{T} \mathbf{B}^* (e_{\alpha} \otimes y) \right\rangle \\
&= \sum_{q=0}^{\infty} \left\langle \sum_{\gamma \in \mathbb{F}_{n_1}^+} \sqrt{b_{\alpha\gamma}} e_{\alpha\gamma} \otimes \Delta_{\mathbf{T}_1} T_{1,\gamma}^* T_{1,\alpha}^* h, \right. \\
&\quad \left. \sum_{\substack{\sigma \in \mathbb{F}_{n_1}^+ \\ 1 \leq |\sigma| \leq k_1}} \left(\sum_{\substack{\alpha_1, \dots, \alpha_q \in \mathbb{F}_{n_1}^+ \\ 1 \leq |\alpha_i| \leq k_1}} \sqrt{a_{\alpha_1} \cdots a_{\alpha_q} a_{\sigma}} \frac{\sqrt{b_{\alpha}}}{\sqrt{b_{\alpha\sigma\alpha_q \cdots \alpha_1}}} e_{\alpha\sigma\alpha_q \cdots \alpha_1} C^* D_{(\alpha_1)}^* \cdots D_{(\alpha_q)}^* B_{(\sigma)}^* y \right) \right\rangle \\
&= \sum_{q=0}^{\infty} \sum_{\substack{\sigma \in \mathbb{F}_{n_1}^+ \\ 1 \leq |\sigma| \leq k_1}} \sum_{\substack{\alpha_1, \dots, \alpha_q \in \mathbb{F}_{n_1}^+ \\ 1 \leq |\alpha_i| \leq k_1}} \left\langle \sqrt{b_{\alpha\sigma\alpha_q \cdots \alpha_1}} e_{\alpha\sigma\alpha_q \cdots \alpha_1} \otimes \Delta_{\mathbf{T}_1} T_{1,\sigma\alpha_q \cdots \alpha_1}^* T_{1,\alpha}^* h, \right. \\
&\quad \left. \sqrt{a_{\alpha_1} \cdots a_{\alpha_q} a_{\sigma}} \frac{\sqrt{b_{\alpha}}}{\sqrt{b_{\alpha\sigma\alpha_q \cdots \alpha_1}}} e_{\alpha\sigma\alpha_q \cdots \alpha_1} C^* D_{(\alpha_1)}^* \cdots D_{(\alpha_q)}^* B_{(\sigma)}^* y \right\rangle \\
&= \sum_{q=0}^{\infty} \sum_{\substack{\sigma \in \mathbb{F}_{n_1}^+ \\ 1 \leq |\sigma| \leq k_1}} \sum_{\substack{\alpha_1, \dots, \alpha_q \in \mathbb{F}_{n_1}^+ \\ 1 \leq |\alpha_i| \leq k_1}} \left\langle \sqrt{b_{\alpha}} \sqrt{a_{\alpha_1} \cdots a_{\alpha_q} a_{\sigma}} \Delta_{\mathbf{T}_1} T_{1,\sigma\alpha_q \cdots \alpha_1}^* T_{1,\alpha}^* h, C^* D_{(\alpha_1)}^* \cdots D_{(\alpha_q)}^* B_{(\sigma)}^* y \right\rangle \\
&= \sum_{q=0}^{\infty} \left\langle \left[\begin{array}{c} \sqrt{a_{\alpha_1} \cdots a_{\alpha_q} a_{\sigma}} \Delta_{\mathbf{T}_1} T_{1,\sigma\alpha_q \cdots \alpha_1}^* \\ \vdots \\ \alpha_1, \dots, \alpha_q \in \mathbb{F}_{n_1}^+ \\ 1 \leq |\alpha_i| \leq k_1 \\ \vdots \\ \sigma \in \mathbb{F}_{n_1}^+, 1 \leq |\sigma| \leq k_1 \end{array} \right] \sqrt{b_{\alpha}} T_{1,\alpha}^* h, \left[\begin{array}{c} C^* D_{(\alpha_1)}^* \cdots D_{(\alpha_q)}^* B_{(\sigma)}^* y \\ \vdots \\ \alpha_1, \dots, \alpha_q \in \mathbb{F}_{n_1}^+ \\ 1 \leq |\alpha_i| \leq k_1 \\ \vdots \\ \sigma \in \mathbb{F}_{n_1}^+, 1 \leq |\sigma| \leq k_1 \end{array} \right] \right\rangle \\
&= \sum_{q=0}^{\infty} \left\langle B \text{diag}_{m_1} (D \text{diag}_{m_1} (\cdots D \text{diag}_{m_1} (C \Delta_{\mathbf{T}_1}) \mathbf{X}_1^* \cdots) \mathbf{X}_1^*) \mathbf{X}_1^* (\sqrt{b_{\alpha}} T_{1,\alpha}^*) h, y \right\rangle.
\end{aligned}$$

Hence and using relation (2.12), we obtain

$$\begin{aligned}
& \langle \varphi_{U^*}(\mathbf{\Lambda}_1)^* K_{f, \mathbf{T}_1} h, e_{\alpha} \otimes y \rangle \\
&= \left\langle \sqrt{b_{\alpha}} \Delta_{\mathbf{T}_1} T_{1,\alpha}^* h, A^* y \right\rangle + \sum_{q=0}^{\infty} \left\langle B \text{diag}_{m_1} (D \text{diag}_{m_1} (\cdots D \text{diag}_{m_1} (C \Delta_{\mathbf{T}_1}) \mathbf{X}_1^* \cdots) \mathbf{X}_1^*) \mathbf{X}_1^* (\sqrt{b_{\alpha}} T_{1,\alpha}^*) h, y \right\rangle.
\end{aligned}$$

Now, using Lemma 2.2, we deduce that

$$\langle \varphi_{U^*}(\mathbf{\Lambda}_1)^* K_{f, \mathbf{T}_1} h, e_{\alpha} \otimes y \rangle = \left\langle \text{diag}_{m_2}(\Delta_{\mathbf{T}'_1}) \begin{bmatrix} \sqrt{c_{\beta}} T_{2,\beta}^* \\ \vdots \\ \beta \in \mathbb{F}_{n_2}^+, 1 \leq |\beta| \leq k_2 \end{bmatrix} (\sqrt{b_{\alpha}} T_{1,\alpha}^*) h, y \right\rangle$$

for any $h \in \mathcal{H}$. Hence, using the definition of the noncommutative Poisson kernel and the fact that $\mathbf{T}_2 \in \mathcal{I}(\mathbf{T}_1, \mathbf{T}'_1)$, we deduce that, for any $\beta \in \mathbb{F}_{n_2}^+$ with $1 \leq |\beta| \leq k_1$, $h \in \mathcal{H}$, and $z \in \mathcal{D}_{\mathbf{T}'_1}$,

$$\begin{aligned}
\langle K_{f, \mathbf{T}'_1} \sqrt{c_{\beta}} T_{2,\beta}^* h, e_{\alpha} \otimes z \rangle &= \left\langle \sum_{k=0}^{\infty} \sum_{\sigma \in \mathbb{F}_{n_1}^+} e_{\sigma} \otimes \sqrt{b_{\sigma}} \Delta_{\mathbf{T}'_1} (T'_{1,\sigma})^* \sqrt{c_{\beta}} T_{2,\beta}^* h, e_{\alpha} \otimes z \right\rangle \\
&= \left\langle \sqrt{b_{\alpha}} \Delta_{\mathbf{T}'_1} (T'_{1,\alpha})^* \sqrt{c_{\beta}} T_{2,\beta}^* h, z \right\rangle = \left\langle \sqrt{b_{\alpha}} \Delta_{\mathbf{T}'_1} \sqrt{c_{\beta}} T_{2,\beta}^* T_{1,\alpha}^* h, z \right\rangle \\
&= \langle \varphi_{U^*}(\mathbf{R})^* K_{\mathbf{T}_1} h, e_{\alpha} \otimes y \rangle = \langle \varphi_{(\beta)}(\mathbf{R})^* K_{\mathbf{T}_1} h, e_{\alpha} \otimes z \rangle,
\end{aligned}$$

where $y = \bigoplus_{\substack{\gamma \in \mathbb{F}_2^{+} \\ 1 \leq |\gamma| \leq k_2}} y_{(\gamma)}$ with $y_{(\gamma)} = 0$ if $\gamma \neq \beta$ and $y_{(\beta)} = z$. Consequently,

$$K_{f, \mathbf{T}_1} T_{2, \beta}^* = \frac{1}{\sqrt{c_\beta}} \varphi_{(\beta)}(\mathbf{\Lambda}_1)^* K_{f, \mathbf{T}_1}, \quad \beta \in \mathbb{F}_{n_2}^+, 1 \leq |\beta| \leq k_2.$$

The last two relations in the theorem are due to relation (1.2) applied to $\mathbf{T}_1 \in \mathcal{D}_f(\mathcal{H})$ and $\mathbf{T}'_1 \in \mathcal{D}_f(\mathcal{H}')$, respectively. The proof is complete. \square

As a consequence of Theorem 2.4, we obtain a new proof for the commutant lifting theorem for the pure elements of the noncommutative domain \mathcal{D}_f (see [15]) as well as a constructive method to obtain the lifting.

Theorem 2.5. *Let $\mathbf{T}_1 := (T_{1,1}, \dots, T_{1,n_1}) \in \mathcal{D}_f(\mathcal{H})$ and $\mathbf{T}'_1 := (T'_{1,1}, \dots, T'_{1,n_1}) \in \mathcal{D}_f(\mathcal{H}')$ be pure tuples of operators and let $\mathbf{W}_1 := [W_{1,1}, \dots, W_{1,n_1}]$ be the universal model associated with the noncommutative domain \mathcal{D}_f . If $A : \mathcal{H}' \rightarrow \mathcal{H}$ is an operator such that*

$$AT'_{1,i} = T_{1,i}A, \quad i \in \{1, \dots, n_1\},$$

then there is an operator $D : F^2(H_{n_1}) \otimes \mathcal{D}_{\mathbf{T}'_1} \rightarrow F^2(H_{n_1}) \otimes \mathcal{D}_{\mathbf{T}_1}$ such that

$$D(W_{1,i} \otimes I_{\mathcal{D}_{\mathbf{T}'_1}}) = (W_{1,i} \otimes I_{\mathcal{D}_{\mathbf{T}_1}})D, \quad i \in \{1, \dots, n_1\},$$

$D^|_{\mathcal{H}} = A^*$, and $\|B\| = \|A\|$, where \mathcal{H} and \mathcal{H}' are identified with co-invariant subspaces of $\{W_{1,i} \otimes I_{\mathcal{D}_{\mathbf{T}}}\}_{i=1}^{n_1}$ and $\{W_{1,i} \otimes I_{\mathcal{D}_{\mathbf{T}'}}\}_{i=1}^{n_1}$, respectively.*

Proof. Without loss of generality, we can assume that $\|A\| = 1$. Since $A \in \mathcal{I}(\mathbf{T}'_1, \mathbf{T}_1)$, we can apply Theorem 2.4 in the particular case when $n_2 = 1$, $\mathbf{T}_2 := A$, and $g = X$. Consequently, there is a contractive multi-analytic operator $\varphi(\mathbf{\Lambda}_1) \in \mathcal{R}_{n_1}^\infty(\mathcal{D}_f) \bar{\otimes} B(\mathcal{D}_{\mathbf{T}'_1}, \mathcal{D}_{\mathbf{T}_1})$ such that $K_{f, \mathbf{T}'_1} A^* = \varphi(\mathbf{\Lambda}_1)^* K_{f, \mathbf{T}_1}$. Since \mathbf{T}_1 and \mathbf{T}'_1 are pure elements, the noncommutative Poisson kernels K_{f, \mathbf{T}_1} and K_{f, \mathbf{T}'_1} are isometries. Under the identifications of \mathcal{H} and \mathcal{H}' with $K_{f, \mathbf{T}_1} \mathcal{H}$ and $K_{f, \mathbf{T}'_1} \mathcal{H}'$, respectively, we have $A^* = \varphi(\mathbf{\Lambda}_1)^*|_{\mathcal{H}}$. Since $1 = \|A^*\| \leq \|\varphi(\mathbf{\Lambda}_1)^*\| \leq 1$, we deduce that $\|A\| = \|\varphi(\mathbf{\Lambda}_1)\|$. Since $D := \varphi(\mathbf{\Lambda}_1)$ intertwines $W_{1,i} \otimes I_{\mathcal{D}_{\mathbf{T}'_1}}$ with $W_{1,i} \otimes I_{\mathcal{D}_{\mathbf{T}_1}}$ for each $i \in \{1, \dots, n_1\}$, the proof is complete. \square

More applications of Theorem 2.4 will be considered in the next sections.

3. NONCOMMUTATIVE VARIETIES, DILATIONS, AND SCHUR REPRESENTATIONS

In this section, we obtain an intertwining dilation theorem on noncommutative varieties in regular domains and a Schur type representation for the unit ball of $\mathcal{R}_n^\infty(\mathcal{V}_J) \bar{\otimes} B(\mathcal{H}', \mathcal{H})$.

First, we recall from [14] and [15] basic facts concerning noncommutative varieties generated by *WOT*-closed two-sided ideals of the Hardy algebra $F_n^\infty(\mathcal{D}_f)$, their universal models, and the Hardy algebras they generate. Let J be a *WOT*-closed two-sided ideal of $F_n^\infty(\mathcal{D}_f)$ such that $J \neq F_n^\infty(\mathcal{D}_f)$. We introduce the noncommutative variety $\mathcal{V}_J(\mathcal{H})$ to be the set of all pure n -tuples $\mathbf{T} := (T_1, \dots, T_n) \in \mathcal{D}_f(\mathcal{H})$ with the property that

$$\varphi(T_1, \dots, T_n) = 0 \quad \text{for any } \varphi \in J,$$

where $\varphi(T_1, \dots, T_n)$ is defined using the $F_n^\infty(\mathcal{D}_f)$ -functional calculus for pure elements in $\mathcal{D}_f(\mathcal{H})$. Define the subspaces of $F^2(H_n)$ by

$$\mathcal{M}_J := \overline{JF^2(H_n)} \quad \text{and} \quad \mathcal{N}_J := F^2(H_n) \ominus \mathcal{M}_J.$$

The subspace \mathcal{N}_J is invariant under the operators W_1^*, \dots, W_n^* and $\Lambda_1^*, \dots, \Lambda_n^*$, and $\mathcal{N}_J \neq 0$ if and only if $J \neq F_n^\infty(\mathcal{D}_f)$. Define the *constrained weighted left* (resp. *right*) *creation operators* associated with the noncommutative variety \mathcal{V}_J by setting

$$B_i := P_{\mathcal{N}_J} W_i|_{\mathcal{N}_J} \quad \text{and} \quad C_i := P_{\mathcal{N}_J} \Lambda_i|_{\mathcal{N}_J}, \quad i = 1, \dots, n.$$

We remark that $\mathbf{B} := (B_1, \dots, B_n)$ is in $\mathcal{V}_J(\mathcal{N}_J)$ and plays the role of universal model for the noncommutative variety \mathcal{V}_J . We will refer to the n -tuples $\mathbf{B} := (B_1, \dots, B_n)$ and $\mathbf{C} := (C_1, \dots, C_n)$ as the

constrained weighted creation operators associated with \mathcal{V}_J . Note that if $J = \{0\}$, then $\mathcal{N}_{\{0\}} = F^2(H_n)$ and $\mathcal{V}_{\{0\}}(\mathcal{H})$ is the set of all pure elements of $\mathcal{D}_f(\mathcal{H})$.

Let $F_n^\infty(\mathcal{V}_J)$ be the WOT-closed algebra generated by B_1, \dots, B_n and the identity and let $R_n^\infty(\mathcal{V}_J)$ be the WOT-closed algebra generated by C_1, \dots, C_n and the identity. We proved in [15] that

$$F_n^\infty(\mathcal{V}_J)' = R_n^\infty(\mathcal{V}_J) \quad \text{and} \quad R_n^\infty(\mathcal{V}_J)' = F_n^\infty(\mathcal{V}_J),$$

where $'$ stands for the commutant. An operator $M \in B(\mathcal{N}_J \otimes \mathcal{K}, \mathcal{N}_J \otimes \mathcal{K}')$ is called multi-analytic with respect to the universal model $\mathbf{B} := (B_1, \dots, B_n)$ if $M(B_i \otimes I_{\mathcal{K}}) = (B_i \otimes I_{\mathcal{K}'})M$ for $i = 1, \dots, n$. We recall that the set of all multi-analytic operators with respect to \mathbf{B} coincides with

$$R_n^\infty(\mathcal{V}_J) \bar{\otimes} B(\mathcal{K}, \mathcal{K}') = P_{\mathcal{N}_J \otimes \mathcal{K}'} [R_n^\infty(\mathcal{D}_f) \bar{\otimes} B(\mathcal{K}, \mathcal{K}')] |_{\mathcal{N}_J \otimes \mathcal{K}}.$$

A similar result holds for the Hardy algebra $F_n^\infty(\mathcal{V}_J)$. Given a noncommutative variety $\mathcal{V}_J(\mathcal{H})$ and $\mathbf{T} \in \mathcal{V}_J(\mathcal{H})$, we define the *constrained Poisson kernel* $K_{J,\mathbf{T}} : \mathcal{H} \rightarrow \mathcal{N}_J \otimes \mathcal{D}_{\mathbf{T}}$ by

$$K_{J,\mathbf{T}} := (P_{\mathcal{N}_J} \otimes I_{\mathcal{D}_{\mathbf{T}}}) K_{J,\mathbf{T}}.$$

We recall that $K_{J,\mathbf{T}}$ is an isometry and satisfies the relation

$$(3.1) \quad K_{J,\mathbf{T}} T_\alpha^* = (B_\alpha^* \otimes I_{\mathcal{D}_{\mathbf{T}}}) K_{J,\mathbf{T}}, \quad \alpha \in \mathbb{F}_n^+.$$

We remark that as a consequence of Theorem 2.5, we deduce the commutant lifting theorem for the elements of the noncommutative varieties \mathcal{V}_J ([15]). More precisely, we can obtain the following result.

Let $J \neq F_{n_1}^\infty(\mathcal{D}_f)$ be a WOT-closed two-sided ideal of the noncommutative Hardy algebra $F_{n_1}^\infty(\mathcal{D}_f)$ and let $\mathbf{B}_1 := (B_{1,1}, \dots, B_{1,n_1})$ and $\mathbf{C}_1 := (C_{1,1}, \dots, C_{1,n_1})$ be the corresponding constrained shifts acting on \mathcal{N}_J . For each $j = 1, 2$, let \mathcal{K}_j be a Hilbert space and $\mathcal{E}_j \subseteq \mathcal{N}_J \otimes \mathcal{K}_j$ be a co-invariant subspace under each operator $B_{1,i} \otimes I_{\mathcal{K}_j}$, $i = 1, \dots, n_1$. If $X : \mathcal{E}_1 \rightarrow \mathcal{E}_2$ is a bounded operator such that

$$X[P_{\mathcal{E}_1}(B_{1,i} \otimes I_{\mathcal{K}_1})|_{\mathcal{E}_1}] = [P_{\mathcal{E}_2}(B_{1,i} \otimes I_{\mathcal{K}_2})|_{\mathcal{E}_2}]X, \quad i = 1, \dots, n_1,$$

then there exists $G(\mathbf{C}_1) \in \mathcal{R}_{n_1}^\infty(\mathcal{V}_J) \bar{\otimes} B(\mathcal{K}_1, \mathcal{K}_2)$ such that

$$G(\mathbf{C}_1)^*|_{\mathcal{E}_2} = X^* \quad \text{and} \quad \|G(\mathbf{C}_1)\| = \|X\|.$$

The analogue of Theorem 2.4 on noncommutative varieties $\mathcal{V}_J(\mathcal{H})$ in the domain $\mathcal{D}_f(\mathcal{H})$ is the following. Recall that $\mathcal{U}_{\mathbf{T}}^{\mathcal{K}}$ is the set of all unitary extensions of the isometry defined by relation (2.2).

Theorem 3.1. *Let $\mathbf{T}_1 := (T_{1,1}, \dots, T_{1,n_1})$ and $\mathbf{T}'_1 := (T'_{1,1}, \dots, T'_{1,n_1})$ be elements of the noncommutative varieties $\mathcal{V}_J(\mathcal{H})$ and $\mathcal{V}_J(\mathcal{H}')$, respectively, and let $\mathbf{T}_2 := (T_{2,1}, \dots, T_{2,n_2}) \in \mathcal{D}_g(\mathcal{H}', \mathcal{H})$ be such that $\mathbf{T}_2 \in \mathcal{I}(\mathbf{T}_1, \mathbf{T}'_1)$. Let $\mathbf{B}_1 := (B_{1,1}, \dots, B_{1,n_1})$ and $\mathbf{C}_1 := (C_{1,1}, \dots, C_{1,n_1})$ be the weighted creation operators associated with the noncommutative variety \mathcal{V}_J . If*

$$\varphi_{U^*}(\mathbf{\Lambda}_1) = [\varphi_{(\beta)}(\mathbf{\Lambda}_1) : \beta \in \mathbb{F}_{n_2}^+, 1 \leq |\beta| \leq k_2]$$

is the contractive multi-analytic operator associated with $U \in \mathcal{U}_{\mathbf{T}}^{\mathcal{K}}$ and \mathbf{T}_1 is pure in $\mathcal{D}_f(\mathcal{H})$, then the following relations hold:

$$K_{J,\mathbf{T}'_1} T_{2,\beta}^* = \frac{1}{\sqrt{c_\beta}} \varphi_{(\beta)}(\mathbf{C}_1)^* K_{J,\mathbf{T}_1}, \quad \beta \in \mathbb{F}_{n_2}^+, 1 \leq |\beta| \leq k_2,$$

$$K_{J,\mathbf{T}_1} T_{1,i}^* = (B_{1,i}^* \otimes I_{\mathcal{D}_{\mathbf{T}_1}}) K_{J,\mathbf{T}_1}, \quad K_{J,\mathbf{T}'_1} (T'_{1,i})^* = (B_{1,i}^* \otimes I_{\mathcal{D}_{\mathbf{T}'_1}}) K_{J,\mathbf{T}'_1}, \quad i \in \{1, \dots, n_1\},$$

where K_{J,\mathbf{T}_1} and K_{J,\mathbf{T}'_1} are the constrained Poisson kernels associated with \mathbf{T}_1 and \mathbf{T}'_1 , respectively.

Proof. Since $\mathbf{T}_1 \in \mathcal{V}_J(\mathcal{H})$ and $\mathbf{T}'_1 \in \mathcal{V}_J(\mathcal{H}')$, the noncommutative Poisson kernels K_{f,\mathbf{T}_1} and K_{f,\mathbf{T}'_1} have ranges in $\mathcal{N}_J \otimes \mathcal{D}_{\mathbf{T}_1}$ and $\mathcal{N}_J \otimes \mathcal{D}_{\mathbf{T}'_1}$, respectively. Due to Theorem 2.4, we have

$$(3.2) \quad K_{f,\mathbf{T}'_1} T_{2,\beta}^* = \frac{1}{\sqrt{c_\beta}} \varphi_{(\beta)}(\mathbf{\Lambda}_1)^* K_{f,\mathbf{T}_1}, \quad \beta \in \mathbb{F}_{n_2}^+, 1 \leq |\beta| \leq k_2.$$

Since \mathcal{N}_J is co-invariant under $\Lambda_{1,1}, \dots, \Lambda_{1,n_1}$, we have

$$\varphi_{(\beta)}(\mathbf{\Lambda}_1)^*(\mathcal{N}_J \otimes \mathcal{D}_{\mathbf{T}_1}) \subset \mathcal{N}_J \otimes \mathcal{D}_{\mathbf{T}'_1}, \quad \beta \in \mathbb{F}_{n_2}^+, 1 \leq |\beta| \leq k_2,$$

and

$$\varphi_{(\beta)}(\mathbf{C}_1) = P_{\mathcal{N}_J \otimes \mathcal{D}_{\mathbf{T}_1}} \varphi_j(\mathbf{\Lambda}_1)|_{\mathcal{N}_J \otimes \mathcal{D}_{\mathbf{T}_1}}, \quad \beta \in \mathbb{F}_{n_2}^+, 1 \leq |\beta| \leq k_2,$$

is a multi-analytic operator with respect to the universal model \mathbf{B} . Note that relation (3.2) implies

$$P_{\mathcal{N}_J \otimes \mathcal{D}_{\mathbf{T}_1'}} K_{f, \mathbf{T}_1'} T_{2, \beta}^* = \frac{1}{\sqrt{c_\beta}} P_{\mathcal{N}_J \otimes \mathcal{D}_{\mathbf{T}_1'}} \varphi_{(\beta)}(\mathbf{\Lambda}_1)^* P_{\mathcal{N}_J \otimes \mathcal{D}_{\mathbf{T}_1}} K_{f, \mathbf{T}_1}, \quad \beta \in \mathbb{F}_{n_2}^+, 1 \leq |\beta| \leq k_2,$$

which proves that

$$K_{J, \mathbf{T}_1} T_{2, \beta}^* = \frac{1}{\sqrt{c_\beta}} \varphi_{(\beta)}(\mathbf{C}_1)^* K_{J, \mathbf{T}_1}, \quad \beta \in \mathbb{F}_{n_2}^+, 1 \leq |\beta| \leq k_2.$$

Since the other relations in the theorem are due to (3.1). The proof is complete. \square

As a consequence of Theorem 3.1 we obtain the following Schur [19] type representation for the unit ball of $\mathcal{R}_{n_1}^\infty(\mathcal{V}_J) \bar{\otimes} B(\mathcal{H}', \mathcal{H})$.

Theorem 3.2. *An operator $\Gamma : \mathcal{N}_J \otimes \mathcal{H}' \rightarrow \mathcal{N}_J \otimes \mathcal{H}$ is in the closed unit ball of $\mathcal{R}_{n_1}^\infty(\mathcal{V}_J) \bar{\otimes} B(\mathcal{H}', \mathcal{H})$ if and only if there is a Hilbert space \mathcal{E} and a unitary operator*

$$\Omega = \begin{bmatrix} E & F \\ G & H \end{bmatrix} : \begin{matrix} \mathcal{H}' \\ \oplus \\ \mathcal{E} \end{matrix} \rightarrow \begin{matrix} \mathcal{H} \\ \oplus \\ \bigoplus_{\substack{\alpha \in \mathbb{F}_{n_1}^+ \\ 1 \leq |\alpha| \leq k_1}} \mathcal{E} \end{matrix}$$

such that $\Gamma = \text{SOT-}\lim_{r \rightarrow 1} \varphi_\Omega(r\mathbf{C}_1)$, where

$$\begin{aligned} \varphi_\Omega(r\mathbf{C}_1) &:= I_{\mathcal{N}_J} \otimes E + (I_{\mathcal{N}_J} \otimes F) \left(I_{\mathcal{N}_J \otimes \mathcal{H}} - \sum_{\substack{\alpha \in \mathbb{F}_{n_1}^+ \\ 1 \leq |\alpha| \leq k_1}} r^{|\alpha|} \sqrt{a_\alpha} C_{1, \bar{\alpha}} \otimes H_{(\alpha)} \right)^{-1} \\ &\quad \times [\sqrt{a_\alpha} C_{1, \bar{\alpha}} \otimes I_{\mathcal{H}} : \alpha \in \mathbb{F}_{n_1}^+, 1 \leq |\alpha| \leq k_1] (I_{\mathcal{N}_J} \otimes G), \end{aligned}$$

where $\mathbf{C}_1 := (C_{1,1}, \dots, C_{1,n_1})$ is the tuple of weighted right creation operators on $F^2(H_{n_1})$ and H has the operator row matrix representation

$$H = \begin{bmatrix} H_{(\alpha)} \\ \vdots \\ \alpha \in \mathbb{F}_{n_1}^+, 1 \leq |\alpha| \leq k_1 \end{bmatrix} : \mathcal{E} \rightarrow \bigoplus_{\alpha \in \mathbb{F}_{n_1}^+, 1 \leq |\alpha| \leq k_1} \mathcal{E}.$$

Proof. Assume that $\Gamma : \mathcal{N}_J \otimes \mathcal{H}' \rightarrow \mathcal{N}_J \otimes \mathcal{H}$ is a contractive multi-analytic operator with respect to the universal model $\mathbf{B}_1 := (B_{1,1}, \dots, B_{1,n_1})$, i.e. $\Gamma(B_{1,i} \otimes I_{\mathcal{H}'}) = (B_{1,i} \otimes I_{\mathcal{H}}) \Gamma$ for any $i \in \{1, \dots, n_1\}$. Due to the commutant lifting theorem for pure elements in \mathcal{D}_f , there exists a contractive multi-analytic operator $\Psi : F^2(H_n) \otimes \mathcal{H}' \rightarrow F^2(H_n) \otimes \mathcal{H}$ with respect to the universal model $\mathbf{W}_1 := (W_{1,1}, \dots, W_{1,n_1})$, i.e. $\Psi(W_{1,i} \otimes I_{\mathcal{H}'}) = (W_{1,i} \otimes I_{\mathcal{H}}) \Psi$ for any $i \in \{1, \dots, n_1\}$, such that $\|\Gamma\| = \|\Psi\|$ and $\Psi^*|_{\mathcal{N}_J \otimes \mathcal{H}} = \Gamma^*$.

Set $\mathbf{T}_1 := (W_{1,1} \otimes I_{\mathcal{H}}, \dots, W_{1,n_1} \otimes I_{\mathcal{H}})$, $\mathbf{T}_1' := (W_{1,1} \otimes I_{\mathcal{H}'}, \dots, W_{1,n_1} \otimes I_{\mathcal{H}'})$, $n_2 = 1$, and $\mathbf{T}_2 := \Psi$. Since $\Psi \in \mathcal{I}(\mathbf{T}_1, \mathbf{T}_1')$, Theorem 2.4 and Lemma 2.3 show that there is a unitary operator

$$\Omega = \begin{bmatrix} E & F \\ G & H \end{bmatrix} : \begin{matrix} \mathcal{D}_{\mathbf{T}_1'} \\ \oplus \\ \mathcal{E} \end{matrix} \rightarrow \begin{matrix} \mathcal{D}_{\mathbf{T}_1} \\ \oplus \\ \bigoplus_{\alpha \in \mathbb{F}_{n_1}^+, 1 \leq |\alpha| \leq k_1} \mathcal{E} \end{matrix}$$

such that $\varphi_\Omega(\mathbf{\Lambda}_1) := \text{SOT-}\lim_{r \rightarrow 1} \varphi_\Omega(r\mathbf{\Lambda}_1)$ is a multi-analytic operator in $\mathcal{R}_{n_1}^\infty(\mathcal{D}_f) \bar{\otimes} B(\mathcal{D}_{\mathbf{T}_1'}, \mathcal{D}_{\mathbf{T}_1})$, where $\varphi_\Omega(r\mathbf{\Lambda}_1)$ is defined as in the theorem and such that $K_{f, \mathbf{T}_1} \Psi^* = \varphi_\Omega(\mathbf{\Lambda}_1)^* K_{f, \mathbf{T}_1}$. Due to relation $I -$

$\sum_{|\beta| \geq 1} a_\beta W_\beta W_\beta^* = P_{\mathbb{C}}$, we deduce that $\mathcal{D}_{\mathbf{T}_1} = \mathcal{H}$ and $\mathcal{D}_{\mathbf{T}'_1} = \mathcal{H}'$. On the other hand, since

$$P_{\mathbb{C}} W_\beta^* e_\alpha = \begin{cases} \frac{1}{\sqrt{b_\beta}} & \text{if } \alpha = \beta \\ 0 & \text{otherwise,} \end{cases}$$

one can easily see that the noncommutative Poisson kernel K_{f, \mathbf{T}_1} is the identity on $F^2(H_{n_1}) \otimes \mathcal{H}$. Consequently, $\Psi = \varphi_\Omega(\mathbf{A}_1)$. Since $\Psi^*|_{\mathcal{N}_J \otimes \mathcal{H}} = \Gamma^*$, we deduce that $\Gamma = \varphi_\Omega(\mathbf{C}_1)$.

To prove the converse, note that, in the particular case when $n_2 = 1$ and $U^* = \Omega$, Lemma 2.3 shows that $\Psi := \text{SOT-lim}_{r \rightarrow 1} \varphi_\Omega(r \mathbf{A}_1)$, where

$$\begin{aligned} \varphi_\Omega(r \mathbf{A}_1) &:= I_{F^2(H_{n_1})} \otimes E + \left(I_{F^2(H_{n_1})} \otimes F \right) \left(I_{F^2(H_{n_1}) \otimes \mathcal{E}} - \sum_{\substack{\alpha \in \mathbb{F}_{n_1}^+ \\ 1 \leq |\alpha| \leq k_1}} r^{|\alpha|} \sqrt{a_\alpha} \Lambda_{1, \tilde{\alpha}} \otimes H_{(\alpha)} \right)^{-1} \\ &\quad \times \left[\sqrt{a_\alpha} \Lambda_{1, \tilde{\alpha}} \otimes I_{\mathcal{H}} : \alpha \in \mathbb{F}_{n_1}^+, 1 \leq |\alpha| \leq k_1 \right] \left(I_{F^2(H_{n_1})} \otimes G \right) \end{aligned}$$

and $H = \begin{bmatrix} H_{(\alpha)} \\ \vdots \\ \alpha \in \mathbb{F}_{n_1}^+, 1 \leq |\alpha| \leq k_1 \end{bmatrix} : \mathcal{E} \rightarrow \bigoplus_{\substack{\alpha \in \mathbb{F}_{n_1}^+ \\ 1 \leq |\alpha| \leq k_1}} \mathcal{E}$, is a contractive multi-analytic operator with respect to \mathbf{W}_1 . Since \mathcal{N}_J is a co-invariant subspace under $\Lambda_{1,1}, \dots, \Lambda_{1,n_1}$, we deduce that $\Gamma := \varphi_\Omega(\mathbf{C}_1) = P_{\mathcal{N}_J \otimes \mathcal{D}_{\mathbf{T}'_1}} \Psi|_{\mathcal{N}_J \otimes \mathcal{D}_{\mathbf{T}_1}}$ is a contractive multi-analytic operator with respect to \mathbf{B}_1 . The proof is complete. \square

4. ANDÔ TYPE DILATIONS AND INEQUALITIES ON NONCOMMUTATIVE BI-DOMAINS AND VARIETIES

In this section, we obtain Andô type dilations and inequalities for the elements of the bi-domain $\mathbf{D}_{(f,g)}$ and a class of noncommutative varieties. The commutative case as well the matrix case are also discussed.

We recall that, given a positive regular formal power series $g = \sum_{\beta \in \mathbb{F}_{n_2}^+, 1 \leq |\beta| \leq k_2} c_\beta X_\beta$, the noncommutative ellipsoid $\mathcal{E}_g(\mathcal{H}) \supseteq \mathcal{D}_g(\mathcal{H})$ is defined by $\mathcal{E}_g(\mathcal{H}) := \left\{ \mathbf{X} := (X_1, \dots, X_{n_2}) : \sum_{|\beta|=1} c_\beta X_\beta X_\beta^* \leq I \right\}$.

One of the most important consequences of the results from Section 2 is the following Andô type dilation for the bi-domain $\mathbf{D}_{(f,g)}(\mathcal{H}) := \mathcal{D}_f(\mathcal{H}) \times_c \mathcal{D}_g(\mathcal{H})$, where f and g are positive regular noncommutative polynomials, and for the noncommutative variety

$$\mathbf{D}_{(f,g)}^J(\mathcal{H}) := \{(\mathbf{T}_1, \mathbf{T}_2) \in \mathbf{D}_{(f,g)}(\mathcal{H}) : \mathbf{T}_1 \in \mathcal{V}_J(\mathcal{H})\}.$$

We recall that $\mathcal{U}_{\mathbf{T}}^K$ is the set of all unitary extensions of the isometry defined by relation (2.2). According to Lemma 2.3, for each $U \in \mathcal{U}_{\mathbf{T}}^K$, the strong operator topology limit $\varphi_{U^*}(\mathbf{A}_1) := \text{SOT-lim}_{r \rightarrow 1} \varphi_{U^*}(r \mathbf{A}_1)$ exists and defines a contractive multi-analytic operator.

Theorem 4.1. *Let $\mathbf{T} = (\mathbf{T}_1, \mathbf{T}_2) \in \mathbf{D}_{(f,g)}^J(\mathcal{H})$ with $\mathbf{T}_1 = (T_{1,1}, \dots, T_{1,n_1})$ and $\mathbf{T}_2 := (T_{2,1}, \dots, T_{2,n_2})$. If*

$$\varphi_{U^*}(\mathbf{A}_1) = (\varphi_{(\beta)}(\mathbf{A}_1) : \beta \in \mathbb{F}_{n_2}^+, 1 \leq |\beta| \leq k_2)$$

is the contractive multi-analytic operator associated with $U \in \mathcal{U}_{\mathbf{T}}^K$, then

$$K_{J, \mathbf{T}_1} T_{1, \alpha}^* T_{2, \beta}^* = (B_{1, \alpha}^* \otimes I_{\mathcal{D}_{\mathbf{T}_1}}) \psi_\beta(\mathbf{C}_1)^* K_{J, \mathbf{T}_1}, \quad \alpha \in \mathbb{F}_{n_1}^+, \beta \in \mathbb{F}_{n_2}^+,$$

where

- (i) $\mathbf{B}_1 := (B_{1,1}, \dots, B_{1,n_1})$ and $\mathbf{C}_1 := (C_{1,1}, \dots, C_{1,n_1})$ are the constrained creation operators associated with the variety \mathcal{V}_J ;
- (ii) K_{J, \mathbf{T}_1} is the constrained Poisson kernel associated \mathcal{V}_J ;
- (iii) $\psi(\mathbf{C}_1) := (\psi_1(\mathbf{C}_1), \dots, \psi_{n_2}(\mathbf{C}_1)) \in \mathcal{E}_g(\mathcal{N}_J \otimes \mathcal{D}_{\mathbf{T}_1})$, where

$$\psi_j(\mathbf{C}_1) := \frac{1}{\sqrt{c_{g_j}}} \varphi_{(g_j)}(\mathbf{C}_1), \quad j \in \{1, \dots, n_2\}.$$

Proof. In the particular case when $\mathbf{T}_1 = \mathbf{T}'_1$, Theorem 3.1 shows that $K_{J,\mathbf{T}_1}T_{2,j}^* = \psi_j(\mathbf{C}_1)^*K_{J,\mathbf{T}_1}$ for $j \in \{1, \dots, n_2\}$ and $K_{J,\mathbf{T}_1}T_{1,i}^* = (B_{1,i}^* \otimes I_{\mathcal{D}_{\mathbf{T}_1}})K_{J,\mathbf{T}_1}$ for $i \in \{1, \dots, n_1\}$. Hence, the relation in the theorem follows. \square

We remark that Theorem 4.1 provides a model and a characterization of the elements $(\mathbf{T}_1, \mathbf{T}_2) \in \mathcal{D}_f(\mathcal{H}) \times_c \mathcal{E}_g(\mathcal{H})$ with $\mathbf{T}_1 \in \mathcal{V}_J(\mathcal{H})$. Indeed, if $\mathbf{T} = (\mathbf{T}_1, \mathbf{T}_2) \in B(\mathcal{H})^{n_1} \times_c B(\mathcal{H})^{n_2}$, then $\mathbf{T} \in \mathcal{D}_f(\mathcal{H}) \times_c \mathcal{E}_g(\mathcal{H})$ with $\mathbf{T}_1 \in \mathcal{V}_J(\mathcal{H})$ if and only if there is a Hilbert space \mathcal{D} , a multi-analytic operator (with respect to \mathbf{B}_1)

$$\psi(\mathbf{C}_1) = (\psi_1(\mathbf{C}_1), \dots, \psi_{n_2}(\mathbf{C}_1)) \in \mathcal{E}_g(\mathcal{N}_J \otimes \mathcal{D}),$$

and a co-invariant subspace $\mathcal{M} \subset \mathcal{N}_J \otimes \mathcal{D}$ under each of the operators $B_{1,i} \otimes I_{\mathcal{D}}$ and $\varphi_j(\mathbf{C}_1)$, where $i \in \{1, \dots, n_1\}$ and $j \in \{1, \dots, n_2\}$, such that \mathcal{M} can be identified with \mathcal{H} ,

$$(B_{1,i}^* \otimes I_{\mathcal{D}})|_{\mathcal{H}} = T_{1,i}^*, \quad \text{and} \quad \varphi_j(\mathbf{C}_1)|_{\mathcal{H}} = T_{2,j}^*.$$

Note that the direct implication is due to Theorem 4.1 under the identification of \mathcal{H} with $K_{J,\mathbf{T}_1}\mathcal{H}$. The converse is obvious.

In what follows, we obtain Andô type inequalities for the bi-domain $\mathbf{D}_{(f,g)}^J(\mathcal{H})$ and the noncommutative variety $\mathbf{D}_{(f,g)}^J(\mathcal{H})$. First, we consider the case when $\mathbf{T}_1 = (T_{1,1}, \dots, T_{1,n_1})$ and $\mathbf{T}_2 := (T_{2,1}, \dots, T_{2,n_2})$ have the property that $\mathbf{T} = (\mathbf{T}_1, \mathbf{T}_2) \in \mathbf{D}_{(f,g)}^J(\mathcal{H})$ with $d_i := \dim \mathcal{D}_{\mathbf{T}_i} < \infty$ and $d_1 + m_1d_2 = m_2d_1 + d_2$, where

$$m_i := \text{card}\{\alpha \in \mathbb{F}_{n_j}^+ : 1 \leq |\alpha| \leq k_j\}, \quad j = 1, 2.$$

The set $\mathcal{U}_{\mathbf{T}}$ consists of unitary extensions $U : \mathcal{D}_{\mathbf{T}_1} \oplus \bigoplus_{\alpha \in \mathbb{F}_{n_1}^+, 1 \leq |\alpha| \leq k_1} \mathcal{D}_{\mathbf{T}_2} \rightarrow \bigoplus_{\beta \in \mathbb{F}_{n_2}^+, 1 \leq |\beta| \leq k_2} \mathcal{D}_{\mathbf{T}_1} \oplus \mathcal{D}_{\mathbf{T}_2}$ of the isometry

$$(4.1) \quad U \left(\Delta_{\mathbf{T}_1} h \oplus \bigoplus_{\alpha \in \mathbb{F}_{n_1}^+, 1 \leq |\alpha| \leq k_1} \sqrt{a_\alpha} \Delta_{\mathbf{T}_2} T_{1,\alpha}^* h \right) := \bigoplus_{\beta \in \mathbb{F}_{n_2}^+, 1 \leq |\beta| \leq k_2} \sqrt{c_\beta} \Delta_{\mathbf{T}_1} T_{2,\beta}^* h, \quad h \in \mathcal{H}.$$

Let $\mathbf{Z} := \langle Z_1, \dots, Z_{n_1} \rangle$ and $\mathbf{Z}' := \langle Z'_1, \dots, Z'_{n_2} \rangle$ be noncommutative indeterminates and assume that $Z_i Z'_j = Z'_j Z_i$ for any $i \in \{1, \dots, n_1\}$ and $j \in \{1, \dots, n_2\}$. We denote by $\mathbb{C} \langle \mathbf{Z}, \mathbf{Z}' \rangle$ the complex algebra of all polynomials in indeterminates Z_1, \dots, Z_{n_1} and Z'_1, \dots, Z'_{n_2} . Note that when $n_1 = n_2 = 1$, then $\mathbb{C} \langle \mathbf{Z}, \mathbf{Z}' \rangle$ coincides with the algebra $\mathbb{C}[z, w]$ of complex polynomials in two variable.

Theorem 4.2. *Let $\mathbf{T} = (\mathbf{T}_1, \mathbf{T}_2) \in \mathbf{D}_{(f,g)}^J(\mathcal{H})$ with $\mathbf{T}_1 = (T_{1,1}, \dots, T_{1,n_1})$ and $\mathbf{T}_2 := (T_{2,1}, \dots, T_{2,n_2})$ such that*

$$d_i := \dim \mathcal{D}_{\mathbf{T}_i} < \infty \quad \text{and} \quad d_1 + m_1d_2 = d_2 + m_2d_1,$$

and let $\mathbf{B}_1 := (B_{1,1}, \dots, B_{1,n_1})$ and $\mathbf{C}_1 := (C_{1,1}, \dots, C_{1,n_1})$ be the constrained weighted creation operators associated with the noncommutative variety \mathcal{V}_J . If $U \in \mathcal{U}_{\mathbf{T}}$, then

$$\| [p_{rs}(\mathbf{T}_1, \mathbf{T}_2)]_k \| \leq \| [p_{rs}(\mathbf{B}_1 \otimes I_{\mathbb{C}^{d_1}}, \psi(\mathbf{C}_1))]_k \|, \quad [p_{rs}]_k \in M_k(\mathbb{C} \langle \mathbf{Z}, \mathbf{Z}' \rangle), k \in \mathbb{N},$$

where $\psi(\mathbf{C}_1) = (\psi_1(\mathbf{C}_1), \dots, \psi_{n_2}(\mathbf{C}_1))$ is uniquely determined by U as in Theorem 4.1 and each $\psi_j(\mathbf{C}_1)$ is a $d_1 \times d_1$ -matrix with entries in the Hardy algebra $\mathcal{R}_{n_1}^\infty(\mathcal{V}_J)$.

Proof. Since $d_1 + m_1d_2 = d_2 + m_2d_1$, the set $\mathcal{U}_{\mathbf{T}}$ of all unitary extensions of the isometry U defined by relation (4.1) is non-empty. Fix any $U \in \mathcal{U}_{\mathbf{T}}$ and apply Theorem 4.1 to $\mathbf{T} = (\mathbf{T}_1, \mathbf{T}_2) \in \mathbf{D}_{(f,g)}^J(\mathcal{H})$ and $U \in \mathcal{U}_{\mathbf{T}}$. Then we deduce that

$$K_{J,\mathbf{T}_1}T_{1,\alpha}^*T_{2,\beta}^* = (B_{1,\alpha}^* \otimes I_{\mathcal{D}_{\mathbf{T}_1}}) \psi_\beta(\mathbf{C}_1)^* K_{J,\mathbf{T}_1}$$

for any $\alpha \in \mathbb{F}_{n_1}^+$ and $\beta \in \mathbb{F}_{n_2}^+$. Consequently, if p is any polynomial in $\mathbb{C} \langle \mathbf{Z}, \mathbf{Z}' \rangle$, we obtain

$$K_{J,\mathbf{T}_1}p(\mathbf{T}_1, \mathbf{T}_2) = p(\mathbf{B}_1 \otimes I_{\mathbb{C}^{d_1}}, \psi(\mathbf{C}_1))K_{J,\mathbf{T}_1}.$$

Since $\mathbf{T}_1 \in \mathcal{V}_J(\mathcal{H})$, the noncommutative Poisson kernel K_{J,\mathbf{T}_1} is an isometry, which implies

$$p(\mathbf{T}_1, \mathbf{T}_2) = K_{J,\mathbf{T}_1}^* p(\mathbf{B}_1 \otimes I_{\mathbb{C}^{d_1}}, \psi(\mathbf{C}_1))K_{J,\mathbf{T}_1}.$$

Now, it is clear that

$$\| [p_{rs}(\mathbf{T}_1, \mathbf{T}_2)]_{k \times k} \| \leq \| [p_{rs}(\mathbf{B}_1 \otimes I_{\mathbb{C}^{d_1}}, \psi(\mathbf{C}_1))]_{k \times k} \|, \quad [p_{rs}]_{k \times k} \in M_k(\mathbb{C} \langle \mathbf{Z}, \mathbf{Z}' \rangle), k \in \mathbb{N}.$$

The proof is complete. \square

Denote by \mathcal{Q}_n^* the set of all formal polynomials of the form $q(\mathbf{Z}, \mathbf{Z}') = \sum a_{\alpha, \beta, \gamma, \sigma} Z_\alpha Z'_\beta (Z'_\sigma)^* Z_\gamma^*$, with complex coefficients, where $\mathbf{Z} := \langle Z_1, \dots, Z_{n_1} \rangle$ and $\mathbf{Z}' := \langle Z'_1, \dots, Z'_{n_2} \rangle$. In what follows, we show that if we drop the conditions $d_i := \dim \mathcal{D}_{\mathbf{T}_i} < \infty$ and $d_1 + m_1 d_2 = d_2 + m_2 d_1$, in Theorem 4.2, we can obtain the following Andô type inequality.

Theorem 4.3. *Let $\mathbf{T} = (\mathbf{T}_1, \mathbf{T}_2) \in \mathbf{D}_{(f,g)}^J(\mathcal{H})$ with $\mathbf{T}_1 = (T_{1,1}, \dots, T_{1,n_1})$ and $\mathbf{T}_2 := (T_{2,1}, \dots, T_{2,n_2})$. If $U \in \mathcal{U}_{\mathbf{T}}^K$, then*

$$\|[q_{rs}(\mathbf{T}_1, \mathbf{T}_2)]_k\| \leq \|[q_{rs}(\mathbf{V}_1, \mathbf{V}_2)]_k\|, \quad [p_{rs}]_{k \times k} \in M_k(\mathcal{Q}_n^*),$$

where $\mathbf{V}_1 := \mathbf{B}_1 \otimes I_{\mathcal{D}_{\mathbf{T}_1}}$ and $\mathbf{V}_2 := \psi(\mathbf{C}_1)$ is uniquely determined by U as in Theorem 4.1.

Proof. The proof uses Theorem 4.1 and is similar to the proof of Theorem 4.2. We shall omit it. \square

For any polynomial $p \in \mathbb{C}\langle \mathbf{Z}, \mathbf{Z}' \rangle$, define $\|p\|_u := \sup \|p(\mathbf{T}_1, \mathbf{T}_2)\|$, where the supremum is taken over all pairs $(\mathbf{T}_1, \mathbf{T}_2) \in \mathcal{D}_f(\mathcal{H}) \times_c \mathcal{E}_g(\mathcal{H})$ and any Hilbert space \mathcal{H} . Then $\|\cdot\|_u$ defines an algebra norm on $\mathbb{C}\langle \mathbf{Z}, \mathbf{Z}' \rangle$. Since the proof is very similar to that of Lemma 2.4 from [16], we omit it. If for $[p_{ij}]_k \in M_k(\mathbb{C}\langle \mathbf{Z}, \mathbf{Z}' \rangle)$, we set

$$\|[p_{ij}]_k\|_{u,k} := \|[p_{ij}]_k\|_u := \sup \|[p_{ij}(\mathbf{T}_1, \mathbf{T}_2)]_k\|,$$

where the supremum is taken over all pairs $(\mathbf{T}_1, \mathbf{T}_2) \in \mathcal{D}_f(\mathcal{H}) \times_c \mathcal{E}_g(\mathcal{H})$ and any Hilbert space \mathcal{H} , we obtain a sequence of norms on the matrices over $\mathbb{C}\langle \mathbf{Z}, \mathbf{Z}' \rangle$. We call $(\mathbb{C}\langle \mathbf{Z}, \mathbf{Z}' \rangle, \|\cdot\|_{u,k})$ the universal operator algebra for the bi-domain $\mathcal{D}_f(\mathcal{H}) \times_c \mathcal{E}_g(\mathcal{H})$.

In what follows, we prove that the abstract bi-domain

$$\mathcal{D}_f \times_c \mathcal{E}_g := \{\mathcal{D}_f(\mathcal{H}) \times_c \mathcal{E}_g(\mathcal{H}) : \mathcal{H} \text{ is a Hilbert space}\}$$

has a universal model $(\mathbf{W}_1 \otimes I_{\ell^2}, \psi(\mathbf{\Lambda}_1))$, where $\mathbf{W}_1 = (W_{1,1}, \dots, W_{1,n_1})$ and $\mathbf{\Lambda}_1 = (\Lambda_{1,1}, \dots, \Lambda_{1,n_1})$ are the weighted left and right creation operators associated with the regular domain \mathcal{D}_f , respectively, and

$$\psi(\mathbf{\Lambda}_1) = (\psi_1(\mathbf{\Lambda}_1), \dots, \psi_{n_2}(\mathbf{\Lambda}_1)) \in \mathcal{E}_g(F^2(H_{n_1}) \otimes \ell^2)$$

is a certain multi-analytic operator with respect to \mathbf{W}_1 .

Theorem 4.4. *There is a multi-analytic operator $\psi(\mathbf{\Lambda}_1) = (\psi_1(\mathbf{\Lambda}_1), \dots, \psi_{n_2}(\mathbf{\Lambda}_1)) \in \mathcal{E}_g(F^2(H_{n_1}) \otimes \ell^2)$ such that*

$$\|[p_{rs}(\mathbf{T}_1, \mathbf{T}_2)]_k\| \leq \|[p_{rs}(\mathbf{W}_1 \otimes I_{\ell^2}, \psi(\mathbf{\Lambda}_1))]_k\|, \quad p_{rs} \in \mathbb{C}\langle \mathbf{Z}, \mathbf{Z}' \rangle,$$

for any $(\mathbf{T}_1, \mathbf{T}_2) \in \mathcal{D}_f(\mathcal{H}) \times_c \mathcal{E}_g(\mathcal{H})$ and any $k \in \mathbb{N}$.

Proof. Given a matrix $[p_{ij}]_k \in M_k(\mathbb{C}\langle \mathbf{Z}, \mathbf{Z}' \rangle)$, we have $\|[p_{ij}]_k\|_u := \sup \|[p_{ij}(\mathbf{T}_1, \mathbf{T}_2)]_k\|$, where the supremum is taken over all pairs $(\mathbf{T}_1, \mathbf{T}_2) \in \mathcal{D}_f(\mathcal{H}) \times_c \mathcal{E}_g(\mathcal{H})$ and any Hilbert space \mathcal{H} . Using a standard argument, one can prove that the supremum is the same if we consider only infinite dimensional separable Hilbert spaces. Since $r\mathbf{T}_1$ is a pure element in $\mathcal{D}_f(\mathcal{H})$ for any $r \in [0, 1)$, it is clear that

$$\|[p_{ij}]_k\|_u = \sup_{\substack{(\mathbf{T}_1, \mathbf{T}_2) \in \mathcal{D}_f \times_c \mathcal{E}_g \\ \mathbf{T}_1 \text{ pure}}} \|[p_{ij}(\mathbf{T}_1, \mathbf{T}_2)]_k\|.$$

Fix $[p_{ij}]_k \in M_k(\mathbb{C}\langle \mathbf{X}, \mathbf{Y} \rangle)$ and choose a sequence $\left\{(\mathbf{T}_1^{(m)}, \mathbf{T}_2^{(m)})\right\}_{m=1}^\infty$ in $\mathcal{D}_f(\mathcal{H}) \times_c \mathcal{E}_g(\mathcal{H})$ with \mathcal{H} separable and $\mathbf{T}_1^{(m)}$ pure element in $\mathcal{D}_f(\mathcal{H})$, and such that

$$(4.2) \quad \|[p_{ij}]_k\|_u = \sup_m \|[p_{ij}(\mathbf{T}_1^{(m)}, \mathbf{T}_2^{(m)})]_k\|.$$

Using Theorem 4.3, in the particular case when $J = \{0\}$, we find, for each $m \in \mathbb{N}$, a multi-analytic operator $\psi^{(m)}(\mathbf{\Lambda}_1) := (\psi_1^{(m)}(\mathbf{\Lambda}_1), \dots, \psi_{n_2}^{(m)}(\mathbf{\Lambda}_1))$ with respect to \mathbf{W}_1 , which belongs to the ellipsoid $\mathcal{E}_g(F^2(H_{n_1}) \otimes \mathbb{C}^{d(m)})$, where $d(m) := \mathcal{D}_{\mathbf{T}_1^{(m)}}$, such that

$$\|[p_{ij}(\mathbf{T}_1^{(m)}, \mathbf{T}_2^{(m)})]_k\| \leq \|[p_{ij}(\mathbf{W}_1 \otimes I_{\mathbb{C}^{d(m)}}, \psi^{(m)}(\mathbf{\Lambda}_1))]_k\|.$$

Consequently, setting $\oplus_{m=1}^{\infty} \mathbf{T}_1^{(m)} := \left(\oplus_{m=1}^{\infty} T_{1,1}^{(m)}, \dots, \oplus_{m=1}^{\infty} T_{1,n_1}^{(m)} \right) \in \mathcal{D}_f(\oplus_{m=1}^{\infty} \mathcal{H})$, relation (4.2) implies

$$\begin{aligned} \| [p_{ij}]_k \|_u &= \| [p_{ij}(\oplus_{m=1}^{\infty} \mathbf{T}_1^{(m)}, \oplus_{m=1}^{\infty} \mathbf{T}_2^{(m)})]_k \| \\ &\leq \| [p_{ij}(\oplus_{m=1}^{\infty} (\mathbf{W}_1 \otimes I_{\mathbb{C}^{d(m)}}), \oplus_{m=1}^{\infty} \psi^{(m)}(\mathbf{A}_1))]_k \| \leq \| [p_{ij}]_k \|_u. \end{aligned}$$

This shows that

$$(4.3) \quad \| [p_{ij}]_k \|_u = \| [p_{ij}(\mathbf{W}_1 \otimes I_{\ell^2}, \zeta(\mathbf{A}_1))]_k \|,$$

where $\zeta(\mathbf{A}_1) = (\zeta_1(\mathbf{A}_1), \dots, \zeta_{n_2}(\mathbf{A}_1)) := \oplus_{m=1}^{\infty} \psi^{(m)}(\mathbf{A}_1) \in \mathcal{E}_g(F^2(H_{n_1}) \otimes \ell^2)$ is a multi-analytic operator with respect to \mathbf{W}_1 . Let $\mathbb{C}_{\mathbb{Q}} \langle \mathbf{Z}, \mathbf{Z}' \rangle$ be the set of all polynomials with coefficients in $\mathbb{Q} + i\mathbb{Q}$, and let $[p_{ij}^{(1)}]_k, [p_{ij}^{(2)}]_k, \dots$ be an enumeration of the set $\{[p_{ij}]_k : p_{ij} \in \mathbb{C}_{\mathbb{Q}} \langle \mathbf{Z}, \mathbf{Z}' \rangle\}$. Due to relation (4.3), for each $s \in \mathbb{N}$, there is a multi-analytic operator $\zeta^{(s)}(\mathbf{A}_1) = (\zeta_1^{(s)}(\mathbf{A}_1), \dots, \zeta_{n_2}^{(s)}(\mathbf{A}_1)) \in \mathcal{E}_g(F^2(H_{n_1}) \otimes \ell^2)$ such that

$$(4.4) \quad \| [p_{ij}^{(s)}]_k \|_u = \| [p_{ij}^{(s)}(\mathbf{W}_1 \otimes I_{\ell^2}, \zeta^{(s)}(\mathbf{A}_1))]_k \|, \quad s \in \mathbb{N}.$$

Define the multi-analytic operator $\Omega_k(\mathbf{A}_1) := \oplus_{s=1}^{\infty} \zeta^{(s)}(\mathbf{A}_1) \in \mathcal{E}_g(F^2(H_{n_1}) \otimes \ell^2)$ and let us prove that

$$(4.5) \quad \| [q_{ij}]_k \|_u = \| [q_{ij}(\mathbf{W}_1 \otimes I_{\ell^2}, \Omega_k(\mathbf{A}_1))]_k \|$$

for any $[q_{ij}]_k \in M_k(\mathbb{C} \langle \mathbf{Z}, \mathbf{Z}' \rangle)$. Note that relation (4.4) implies

$$(4.6) \quad \| [p_{ij}^{(s)}]_k \|_u = \| [p_{ij}^{(s)}(\mathbf{W}_1 \otimes I_{\ell^2}, \Omega_k(\mathbf{A}_1))]_k \| \quad \text{for any } s \in \mathbb{N}.$$

Fix $[q_{ij}]_k \in M_k(\mathbb{C} \langle \mathbf{Z}, \mathbf{Z}' \rangle)$ and $\epsilon > 0$, and choose $[p_{ij}^{(s_0)}]_k$ such that

$$(4.7) \quad \| [q_{ij}]_k - [p_{ij}^{(s_0)}]_k \|_u < \epsilon.$$

Using relations (4.3), (4.7), and (4.6), we deduce that there is $\zeta^{(q)} := (\zeta_1^{(q)}(\mathbf{A}_1), \dots, \zeta_{n_2}^{(q)}(\mathbf{A}_1))$ in the ellipsoid $\mathcal{E}_g(F^2(H_{n_1}) \otimes \ell^2)$ such that

$$\begin{aligned} \| [q_{ij}]_k \|_u &= \| [q_{ij}(\mathbf{W}_1 \otimes I_{\ell^2}, \zeta^{(q)}(\mathbf{A}_1))]_k \| \leq \| [p_{ij}^{(s_0)}(\mathbf{W}_1 \otimes I_{\ell^2}, \zeta^{(q)}(\mathbf{A}_1))]_k \| + \epsilon \\ &\leq \| [p_{ij}^{(s_0)}]_k \|_u + \epsilon = \| [p_{ij}^{(s_0)}(\mathbf{W}_1 \otimes I_{\ell^2}, \Omega_k(\mathbf{A}_1))]_k \| + \epsilon \\ &\leq \| [q_{ij}(\mathbf{W}_1 \otimes I_{\ell^2}, \Omega_k(\mathbf{A}_1))]_k \| + 2\epsilon \end{aligned}$$

for any $\epsilon > 0$, which proves relation (4.5). Note that $\psi(\mathbf{A}_1) := \oplus_{k=1}^{\infty} \Omega_k(\mathbf{A}_1)$ is a multi-analytic operator which belongs to the ellipsoid $\mathcal{E}_g(F^2(H_{n_1}) \otimes \ell^2)$ and $\| [q_{ij}]_k \|_u = \| [q_{ij}(\mathbf{W}_1 \otimes I_{\ell^2}, \psi(\mathbf{A}_1))]_k \|$ for any $[q_{ij}]_k \in M_k(\mathbb{C} \langle \mathbf{Z}, \mathbf{Z}' \rangle)$ and any $k \in \mathbb{N}$. The proof is complete. \square

Theorem 4.4 shows that $(\mathbb{C} \langle \mathbf{Z}, \mathbf{Z}' \rangle, \|\cdot\|_{u,k})$ can be realized completely isometrically isomorphic as a concrete algebra of operators. The closed non-self-adjoint algebra generated by the operators $W_{1,1} \otimes I_{\ell^2}, \dots, W_{1,n_1} \otimes I_{\ell^2}, \psi_1(\mathbf{A}_1), \dots, \psi_{n_2}(\mathbf{A}_1)$ and the identity is denoted by $\mathcal{A}(\mathcal{D}_f \times_c \mathcal{E}_g)$ and can be seen as the universal operator algebra of the bi-domain $\mathcal{D}_f \times_c \mathcal{E}_g$.

We remark that the noncommutative variety $\mathcal{V}_J \times_c \mathcal{E}_g$ also has a universal model. Similarly to the proof of Theorem 4.4, one can show that there is a multi-analytic operator $\psi(\mathbf{C}_1) = (\psi_1(\mathbf{C}_1), \dots, \psi_{n_2}(\mathbf{C}_1))$, with respect to \mathbf{B}_1 , in $\mathcal{E}_g(\mathcal{N}_J \otimes \ell^2)$ such that

$$\| [p_{rs}(\mathbf{T}_1, \mathbf{T}_2)]_k \| \leq \| [p_{rs}(\mathbf{B}_1 \otimes I_{\ell^2}, \psi(\mathbf{C}_1))]_k \|, \quad p_{rs} \in \mathbb{C} \langle \mathbf{Z}, \mathbf{Z}' \rangle,$$

for any $(\mathbf{T}_1, \mathbf{T}_2) \in \mathcal{V}_J(\mathcal{H}) \times_c \mathcal{E}_g(\mathcal{H})$ and any $k \in \mathbb{N}$.

In the end of this section, we discuss the commutative case. Let J_c be the *WOT*-closed two-sided ideal of the Hardy algebra $F_{n_1}^{\infty}(\mathcal{D}_f)$ generated by the commutators $W_j W_i - W_i W_j$ for $i, j \in \{1, \dots, n_1\}$. Note that the variety $\mathcal{V}_{J_c}(\mathcal{H})$ consists of all pure tuples $(X_1, \dots, X_{n_1}) \in \mathcal{D}_f(\mathcal{H})$ with commuting entries. The Hardy algebra $F_{n_1}^{\infty}(\mathcal{V}_{J_c})$ is the *WOT*-closed algebra generated by the compressions $L_i := P_{F_s^2(\mathcal{D}_f)} W_i|_{F_s^2(\mathcal{D}_f)}$, $i = 1, \dots, n_1$, and the identity, where $F_s^2(\mathcal{D}_f) = \mathcal{N}_{J_c}$ is the symmetric weighted Fock

space associated with the noncommutative domain \mathcal{D}_f . In [15], we prove that $F_s^2(\mathcal{D}_f)$ can be identified with a Hilbert space $H^2(\mathcal{D}_f^\circ(\mathbb{C}))$ of holomorphic functions defined on the scalar domain

$$\mathcal{D}_f^\circ(\mathbb{C}) := \left\{ (\lambda_1, \dots, \lambda_{n_1}) \in \mathbb{C}^{n_1} : \sum_{|\alpha| \geq 1} a_\alpha |\lambda_\alpha|^2 < 1 \right\},$$

namely, the reproducing kernel Hilbert space with reproducing kernel $\kappa_f : \mathcal{D}_f^\circ(\mathbb{C}) \times \mathcal{D}_f^\circ(\mathbb{C})$ defined by $\kappa_f(\mu, \lambda) := \frac{1}{1 - \sum_{|\alpha| \geq 1} a_\alpha \mu_\alpha \bar{\lambda}_\alpha}$ for $\mu, \lambda \in C$. We also identified the algebra of all multipliers of the Hilbert space $H^2(\mathcal{D}_f^\circ(\mathbb{C}))$ with the Hardy algebra $F_{n_1}^\infty(\mathcal{V}_{J_c})$. Under this identification, L_i is the multiplier M_{λ_i} by the coordinate function. We denote $\mathbf{M}_{\lambda, n_1} := (M_{\lambda_1}, \dots, M_{\lambda_{n_1}})$. Similarly, one can identify the Hardy algebra $R_{n_1}^\infty(\mathcal{V}_{J_c})$ with the algebra of all multipliers of the Hilbert space $H^2(\mathcal{D}_{\tilde{f}}^\circ(\mathbb{C}))$, where $\tilde{f} := \sum_{|\alpha| \geq 1} a_{\bar{\alpha}} Z_\alpha$. Note also that $\mathcal{D}_f^\circ(\mathbb{C}) = \mathcal{D}_{\tilde{f}}^\circ(\mathbb{C})$.

Theorem 4.5. *Let $f \in \mathbb{C}\langle \mathbf{Z} \rangle$ and $g \in \mathbb{C}\langle \mathbf{Z}' \rangle$ be two positive regular noncommutative polynomials and let $(\mathbf{T}_1, \mathbf{T}_2) \in \mathcal{D}_f(\mathcal{H}) \times_c \mathcal{D}_g(\mathcal{H})$ be such that each tuple $\mathbf{T}_j = (T_{j,1}, \dots, T_{j,n_j})$ has commuting entries and $d_j := \text{rank } \Delta_{\mathbf{T}_j}$, $j = 1, 2$. Then there exist multipliers M_{Φ_f} and M_{Φ_g} of $H^2(\mathcal{D}_f^\circ) \otimes \mathbb{C}^{d_1}$ and $H^2(\mathcal{D}_g^\circ) \otimes \mathbb{C}^{d_2}$, respectively, such that $M_{\Phi_f} \in \mathcal{E}_f(H^2(\mathcal{D}_f^\circ))$, $M_{\Phi_g} \in \mathcal{E}_g(H^2(\mathcal{D}_g^\circ))$, and*

$$\|[p_{rs}(\mathbf{T}_1, \mathbf{T}_2)]_k\| \leq \min \{ \|[p_{rs}(\mathbf{M}_{\lambda, n_1} \otimes I_{\mathbb{C}^{d_1}}, M_{\Phi_f})]_k\|, \|[p_{rs}(M_{\Phi_g}, \mathbf{M}_{\lambda, n_2} \otimes I_{\mathbb{C}^{d_2}})]_k\| \}$$

for any matrix $[p_{rs}]_k \in M_k(\mathbb{C}\langle \mathbf{Z}, \mathbf{Z}' \rangle)$ and any $k \in \mathbb{N}$.

Proof. Applying Theorem 4.3 to the pairs $(\mathbf{T}_1, \mathbf{T}_2) \in \mathbf{D}_{(f,g)}^{J_c}(\mathcal{H})$ and $(\mathbf{T}_2, \mathbf{T}_1) \in \mathbf{D}_{(g,f)}^{J_c}(\mathcal{H})$ and using the the identifications preceding this theorem, one can easily complete the proof. \square

We should mention that all the results of the present paper concerning Andô type dilations and inequalities can be written in the commutative multivariable setting of Theorem 4.5. Moreover, in the particular case when $n_1 = n_2 = 1$, we obtain extensions of Andô's results [2], Agler-McCarthy's inequality [1], and Das-Sarkar extension [6], to larger classes of commuting operators.

A few remarks concerning the matrix case when $n_1 = n_2 = 1$ are necessary. If T_1 and T_2 are commuting contractive matrices with no eigenvalues of modulus 1, Agler and McCarthy proved, in their remarkable paper [1], that the pair (T_1, T_2) has a co-isometric extension (M_z^*, M_Φ^*) on $H^2 \otimes \mathbb{C}^d$ and, for any polynomial p in two variables,

$$\|p(T_1, T_2)\| \leq \|p(M_z \otimes I_{\mathbb{C}^d}, M_\Phi)\| \leq \|p\|_V,$$

where V is a distinguished variety in \mathbb{D}^2 depending on T_1 and T_2 .

Let $f \in \mathbb{C}[z]$ be a positive regular polynomial in one variable and let $T \in \mathcal{D}_f(\mathbb{C}^n)$ be an $n \times n$ matrix which is pure with respect to the regular domain \mathcal{D}_f , i.e. $\text{SOT-}\lim_{m \rightarrow \infty} \Phi_{f, \mathbf{T}}^m(I) = 0$. Let $m_T(z) = (z - \lambda_1)^{n_1} \dots (z - \lambda_k)^{n_k}$ be the minimal polynomial of T and let J_{m_T} be the WOT -closed two sided ideal of the Hardy algebra $F_1^\infty(\mathcal{D}_f)$ generated by $m_T(\mathbf{S})$, where \mathbf{S} is the weighted shift associated with the domain \mathcal{D}_f . Note that the variety $\mathcal{V}_{J_{m_T}}(\mathbb{C}^n)$ consists of all $n \times n$ matrices X such that $m_T(X) = 0$. On the other hand, $B := P_{\mathcal{N}_{J_{m_T}}} \mathbf{S}|_{\mathcal{N}_{J_{m_T}}}$ is the universal model of the variety $\mathcal{V}_{J_{m_T}}$, and the ellipsoid $\mathcal{E}_f(\mathbb{C}^n)$ is a matrix-valued ball. In this case, the analytic operators with respect to B are the elements $\varphi(B)$ of the Hardy algebra $R_1^\infty(\mathcal{V}_{J_{m_T}})$.

Theorem 4.6. *Let T_1 and T_2 be commuting matrices which are pure elements in $\mathcal{D}_f(\mathbb{C}^n)$ and $\mathcal{D}_g(\mathbb{C}^n)$ and let B_1 and B_2 be their universal models, respectively. If $d_j := \dim(I - T_j T_j^*)^{1/2} \mathcal{H}$, $j = 1, 2$, then there exist matrix-valued analytic operators $\varphi_1(B_1) \in \mathcal{E}_f(\mathcal{N}_{J_{m_{T_1}}} \otimes \mathbb{C}^{d_1})$ with respect to B_1 and $\varphi_2(B_2) \in \mathcal{E}_g(\mathcal{N}_{J_{m_{T_2}}} \otimes \mathbb{C}^{d_2})$ with respect to B_2 , such that*

$$\|[p_{rs}(T_1, T_2)]_k\| \leq \min \{ \|[p_{rs}(B_1 \otimes I_{\mathbb{C}^{d_1}}, \varphi_1(B_1))]_k\|, \|[p_{rs}(\varphi_2(B_2), B_2 \otimes I_{\mathbb{C}^{d_2}})]_k\| \},$$

for any $[p_{rs}]_k \in M_k(\mathbb{C}[z, w])$.

Proof. Applying Theorem 4.3 to the pairs $(T_1, T_2) \in \mathbf{D}_{(f,g)}^{J_{mT_1}}(\mathcal{H})$ and $(T_2, T_1) \in \mathbf{D}_{(g,f)}^{J_{mT_2}}(\mathcal{H})$, the result follows. \square

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